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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

CLASSIFICATION OF IRRIGABLE LANDS

By W. W. Johnston,[1] Esq.

Synopsis

Land classification as conducted in the Bureau of Reclamation is on the basis of utility for irrigation farming. Three main types are made which are similar in main essentials but differ in detail and refinement. The principal function of all these is to eliminate from the area to be irrigated (arable land) lands which do not have sufficient productive capacity to support a farm family at a reasonable level of living and permit the payment of construction assessments. The "arable" lands are further divided into classes and subclasses on the basis of productive value under irrigation.

Introduction

The term "Land Classification," as used by various agencies in recent years, has been applied to a variety of land maps that differ widely both in purpose and detail. In some cases it means the separation of broad areas on the basis of "type of use," such as industrial areas, recreational areas, and the like, as distinguished from lands that are primarily suited to agricultural uses. More commonly, lands are classified on the basis of their adaptation to different types of agricultural use, such as forest areas, grazing lands, and lands suited to different types of crop production.

Land classification, as conducted in the Bureau of Reclamation and discussed in this paper, is on the basis of utility for irrigation farming, and much of its strength lies in this singleness of purpose. The most important phase of this work is the separation of lands suitable for irrigation (termed "arable") from inferior lands not suited to irrigation development (non-arable). To be suitable for irrigation, land must not only be located under the ditch and of a surface relief even enough to be irrigated, but it also must have sufficient productive capacity to support a family at a reasonable level of living and to permit the payment of irrigation assessments. The arable lands are divided further into two or three classes on the basis of productive value and the ability

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to meet irrigation assessments, class 1 representing the highest class arable land and class 3 the lowest. The non-arable lands are placed in class 6 except where special classes are mapped that include portions of the "non-arable" land as here defined, on which surplus waters may be used to advantage on a yearly rental basis for some limited use, such as irrigating pasture. Frequently, subsequent economic investigations will indicate that when intermingled with better land, so that a farm unit will include only a small percentage of this type, such portions of the acreage in the special classes properly may be supplied with a regular water right and be subject to regular irrigation assessments. In reclassification of old projects, class 5, a suspended class, is frequently used to include lands that are temporarily unproductive, pending the construction of needed drainage or other action which will restore productivity.

A study of its history shows that since its inception there has always been some kind of land classification practiced by the Bureau of Reclamation. The earlier land studies were generally not extended to great detail, and the importance of soil conditions was not recognized to the same extent as at present. Little was known about arid soils and their response to irrigation, and in the earlier years there was a tendency on the part of practically every one interested in irrigation to believe that all land below the irrigation ditch was susceptible of producing satisfactory crops if only the water were delivered to it. Experience has demonstrated, however, that good soil is as important as sufficient water, and few irrigation developments have stabilized with as much land under irrigation as was originally contemplated.

The Modern System of Land Classification

The modern system of classifying irrigable lands began with the passage of the so-called "Fact-Finders Act" of December 9, 1924, wherein provision was made for the classification of project lands on the basis of ability to meet construction charges. Experience and study during the intervening years have resulted in the development of methods and standards now in use. Although land classification is used for essentially economic purposes, it is necessary to base the distinction between classes on physical conditions, of which soil, topographic, and drainage characteristics are of primary importance. Climate is likewise an important factor in gaging the productive capacity of an area, but generally it is reasonably uniform within the boundaries of a single project and therefore does not enter into the division of land into classes.

Soil characteristics vary so greatly from place to place that it is difficult to establish a set of soil conditions that are uniformly applicable as a guide in determining irrigability. In the central valleys of California, for instance, medium to light-textured sandy loams and loams of recent alluvial deposition, such as river deposits adjoining the Sacramento River and soils formed from alluvial fan deposits of the smaller streams, are the more successfully irrigated, whereas the older alluvial soils have developed hardpans and in many cases have been leached too extensively to be highly productive or to respond well to irrigation. In eastern Montana, on the other hand, the soils of mature development, which have become well weathered and have developed definite profile characteristics, are usually the most successful irrigated lands; and here

the soils of recent alluvial deposition are likely either to be heavy intractable clays that have been formed from materials washed in from "raw" shale hills, or sandy soils derived from sandstone alluvium that has been little altered and is of limited fertility.

Certain soil characteristics are uniformly necessary, however. To be satisfactory for irrigation development, the soil must have a reasonably high water-holding capacity; it must be readily penetrable by water and still have a slow enough infiltration rate so that it can be irrigated without excessive percolation losses; it must be free from "black alkali" or a sodium-saturated condition; and it must be either nonsaline or of a type from which the moderate amounts of saline salts that it may contain can be leached readily. Also, the soil must be sufficiently deep to allow necessary root development and permit soil drainage. Arid and semi-arid soils that meet these requirements are practically always well supplied with mineral plant nutrients. The quantity of nitrogen is generally deficient, but this element is readily maintained by the production of leguminous crops that are the backbone of irrigation agriculture.

Topographic considerations are of two main types: First, the degree of slope; and second, the character of the surface relief—whether it is smooth or uneven. Likewise, the characteristics of the soil in respect to erosion, permeability to water, and susceptibility to "subbing" or lateral movement of water between furrows must be considered, and the methods of irrigation that have proved to be adapted to the region cannot be ignored in fixing the topographic limits for the different classes. In the broad smooth valleys of Arizona and California, the use of large heads of water and flooding in borders, checks, or basins has proved to be the most practicable method of irrigation for all except row crops and possibly orchards. In central Washington, on the other hand, the corrugation method of irrigation has proved to be well adapted to soil and topographic conditions and is used for all types of crops, almost to the exclusion of other methods. Irrigation heads are small and, particularly on the steeper slopes, the use of flumes and irrigation pipe, which permit accurate control of water to the corrugation, is common practice. Under the latter conditions slopes of from 10% to 15% are used successfully under irrigation and are classified as arable where soil conditions are favorable. Under conditions prevailing in the former localities, on the other hand, the permissible limit of slope is generally less than 10%.

Lands of moderate gradient, but of uneven surface, are considered both from the standpoint of the cost of necessary leveling and the probable effect of surface soil removal on fertility. In general, deep soils of recent alluvial deposition can be graded heavily with only temporary curtailment of productive capacity, but older soils that have developed zones of lime accumulation in the subsoil, and also soils of limited depth to rock or gravel, cannot be heavily graded without seriously decreasing crop production in spots where heavy cuts have been made.

Some future drainage requirements are to be anticipated on all new projects, and it is difficult, and in many cases impossible, to determine in advance where seepage development will take place. In land classification, therefore, eliminations from the irrigable area because of adverse drainage conditions are limited

to areas without surface outlets and to narrow draws and valleys where early seepage development is to be anticipated and drainage costs will obviously be in excess of benefits. Lands are placed in class 2 or 3 because of adverse drainage conditions where more than average hazards of seepage development exist but conditions are favorable for artificial drainage construction at reasonable cost.

Methods

In order to achieve the objectives of land classification, as previously discussed, and to assure uniform work where a number of classifiers are employed, it has been found necessary to establish certain physical standards or specifications for each class. This must be done for each project studied, although standards used in different parts of the West include certain similarities, and those applying to two areas having similar soils, climate, and irrigation practices will be correspondingly similar. In so far as conditions can be duplicated, standards are based on actual performance of similar lands under irrigation, and to this end it is customary, when establishing classification standards for an area of new land, to study carefully the results that are being obtained under similar soil and topographic conditions in near-by developed areas.

For instance, in starting the classification of Columbia Basin project lands, it was important to know which sandy soils of the area were suitable for irrigation development. This problem was studied on three near-by irrigation districts where sandy texture has been the principal limiting factor in the success of the area under irrigation. Soils were examined on farms that have been farmed profitably under irrigation for a long period of years, on others that have been less successful, and on farms that have gone out of cultivation after a few years' trial. The history of these farms was obtained from well-informed local people, the record of water use was secured, and soil samples were taken for mechanical analyses. The information obtained, considered in the light of the amounts of irrigation water that can be supplied economically, made it possible to fix the standard of soil texture for each class on the basis of demonstrated experience. The standards with respect to topography, stoniness, depth to rock or gravel, and other factors were determined in the same way, although it was necessary to consider irrigated lands in the Yakima and Kittitas valleys, in addition to projects immediately adjoining the Columbia River, in order to duplicate all of the principal soil and topographic conditions encountered on the Columbia Basin project.

The methods followed in making a land classification vary somewhat, depending on the detail required in the particular study. In all cases, however, the boundaries between classes are drawn on the maps in the field after careful examination of all physical conditions, including subsoil examinations. The reason for placing land in a lower class than "1" is indicated by placing an appropriate letter after the classification number. The letter "S" indicates a soil deficiency; "R" loose rock in the plow zone; "T" a topographic deficiency; and "D" a deficiency in drainage characteristics. Two or more letters are used when more than one factor is involved. Thus, the classification as 2ST indicates that the land is inferior to class 1 in both soil and topographic characteristics, whereas an area marked 2T is equal to class 1 in soil, drainage, and

freedom from loose rock in the plow zone, but it is either too rough in surface or too steep in slope to be first class. The boundaries between lands placed in the same class for different reasons are delineated on the land-classification maps, as well as the boundaries between classes. In the ordinary procedure, only the acreage in each class as a whole is measured and reported. The maps include the necessary data, however, to permit the measurement of areas in the several subclasses as well.

Where detailed topographic maps or good aerial photographs are available for use as base maps, it is unnecessary to survey the actual boundaries because with such maps a competent man can sketch the boundaries between classes with no greater error than his judgment in determining where the class boundary should be. Where base maps of this kind are not available, it is necessary to use a plane table. Sufficient test pits or auger borings are made to explore subsoil conditions carefully, these being ordinarily extended to depths of 5 ft. The minimum number for detailed studies is one for each 40 acres; for semi-detailed classification, one for each 160 acres; and enough in reconnaissance studies to meet the less exacting requirements of that kind of study. Alkali determinations of pertinent soils samples are made where the need is indicated, and in some of the larger and more detailed studies field laboratories are maintained where mechanical analyses, percolation studies, and other soil tests can be made.

Types

In recent years the Bureau of Reclamation has attempted to standardize on three main types of land classification that are similar in the main essentials but differ in the detail and degree of accuracy attained. These are "Reconnaissance Land Classification," "Semi-Detailed Land Classification," and "Detailed Land Classification." It has been possible to standardize quite definitely in the case of the last two types, but the degree of accuracy attained in reconnaissance classifications has varied somewhat from one area to another, depending on the particular requirements of the investigation.

Reconnaissance Land Classification

The more general type of reconnaissance is usually made on maps having a scale of 2,000 ft to the inch or on aerial photographs of whatever scale available. This type is applicable to the study of large areas where it is desired to secure general information and to determine the location and extent of areas that are sufficiently promising to warrant more detailed investigations. An example is found in studies made in southwestern Wyoming in connection with the Colorado River investigations. Inspection showed this area to be made up largely of extremely shallow soil but to include also bodies of land of variable extent made up of better soil Consequently, it was decided to make a general reconnaissance land classification for the primary purpose of locating areas of sufficient promise to warrant more detailed study. In the absence of better base maps, township sheets with horizontal scales of 2,000 ft to the inch were made up from township data, and section corners were located and flagged on the exterior boundaries of each township. With these corners for initial control

and a plane table for use in maintaining orientation and location, the general boundaries of class 1 and class 2 areas were mapped in each township, this work requiring an average of four days per township for two men. Arable areas having sufficient size to warrant further consideration, as indicated in the reconnaissance, were then covered by a semi-detailed land classification, this being the type that was followed more generally in these investigations. The results were satisfactory and the cost was a fraction of the amount which would have been required had the entire area been covered in the usual detail.

An example of a more accurate reconnaissance is the investigation the Bureau of Reclamation is making in the Sacramento Valley. This work, which eventually will cover several million acres, is for the purpose of obtaining an inventory of lands that may be irrigated from the Shasta reservoir and from storage which may be developed on tributary streams. Since the complete consummation of plans to be developed as a result of the investigations will stretch far into the future, the present requirement is for information on the location of lands of different capabilities and an estimate of acreage that is reliable for good-sized blocks but not necessarily applicable to a single farm or individual 40-acre tract. It is planned that a more detailed land classification will be made of any project resulting from these and other preliminary investigations before starting construction.

In this study, standards in respect to soil conditions are defined in terms of soil types as mapped in the county soil surveys, but all of the area is examined and the features of the classification are delineated on aerial photographs in the field. Where recent soil surveys are available, little additional work is necessary for the better soil types except to map topographic features and give due consideration to alkali. For inferior soil types, however, where only the more favorable phases are suitable for irrigation development, it is necessary to put down a reasonable number of soil pits and to give careful consideration to soil changes. This has been found to be true with all soil types in areas not covered by recent soil surveys.

The good valley lands that are suited to a wide variety of crops are included in classes 1 and 2. Class 1 includes the better lands which have deep, well-drained soils that are free from harmful quantities of alkaline or saline salts and are of favorable topography. Class 2 includes areas which are still "good farm lands" but are distinctly lower in productive capacity than class 1 because of some limitation such as heavy texture, limited soil depth, moderate salinity, or somewhat unfavorable topography. Class 3 is a "special" class made up of lands having restricted utility for irrigation farming. Class 3a includes the deeper and more productive phases of the "hardpan lands," and class 3b comprises the heavy clays, which have proved suitable for growing rice or for irrigated pasture but are not suited to a wide variety of crops. Portions of the rolling foothills that have irrigation possibilities are likewise in "special" classes, being designated as H-1 or H-2, the first representing the best and the second representing the less desirable of the hill lands that are found to be suitable in some capacity for irrigation.

Class 1 and class 2 lands include those which are capable of continued production and which will give a fair return to the operator, thus permitting the

payment of substantial irrigation assessments. Lands in the "special" classes are types that have been proved by experience to be capable of some production under irrigation; but their development as irrigated land is more hazardous from the standpoint of the man who may settle on them and also the agency that must collect the construction costs. Areas found to be unsuited for irrigation, even in the limited capacity of the special classes, are classed as non-arable (class 6).

Semi-Detailed Land Classification

The semi-detailed land classification (sometimes called detailed reconnaissance) is generally made on maps having a scale of 1,000 ft = 1 in., and is the type most usually followed in areas under investigation for development as individual irrigation projects. Ordinarily, only two classes of arable land (class 1 and class 2) and one class of non-arable land (class 6) are mapped, but special classes are sometimes included when the requirement is indicated. This type is very similar to the "Detailed Land Classification" except that, as the name implies, it is not carried to the same degree of refinement. With this limitation, the discussion of the latter type that follows in later paragraphs will apply to the semi-detailed type, and a thorough description is omitted from this section in the interest of brevity. A set of classification standards, which was used in the investigation of one project (Kendrick project, Wyoming), follows:

Semi-Detailed Land Classification

Class 1, Arable.—

Soil.—

Depth.—Four feet or more to shale or hard sandstone, 2 ft or more to gravel or loose rock.

Texture.—Fine sandy loam to silt loam and not seriously compacted.

Alkali.—Free from more than slight indication of neutral salts (white alkali); free of indication of black alkali either present or prospective.

Topography.—

Smooth slopes of 5% or less with reasonably good-sized areas sloping in the same plane.

Drainage Conditions.—

Soil and topographic conditions such that seepage would not be expected to develop soon and artificial drainage would be feasible at moderate cost.

Class 2, Arable.—(Areas placed in class 2 on account of deficiencies in either soil, topography, or drainage with the following minimum standards):

Soil.—

Depth.—Three feet or more to shale, hard sandstone, or impervious clay; 1½ ft or more to gravel or loose rock.

Texture.—Sandy loam to clay loam—some compaction permitted.

Alkali.—Slight to moderate indication of white alkali but not enough to prevent crop growth; free of indication of black alkali.

Topography.—

Smooth slopes between 5% and 10%, or rougher slopes that may be less than 5% in general gradient.

Drainage Conditions.—

Soil and topographic conditions such that seepage may be expected to develop rather quickly, but with reclamation by artificial drainage at moderate cost appearing feasible.

Class 6, Non-Arable.—(Areas placed in non-arable class because of failure to meet the minimum requirements for class 2 in either soil, topography, or drainage, or because of combinations of such deficiencies, including the following):

Soil.—

Depth.—Shallow soils usually overlying shale.

Texture and Alkali.—Compact clay soils that are generally somewhat alkaline and are only slightly permeable to water; excessively sandy soils that are generally more or less blown up into dunes and depressions.

Topography.—

Areas with rough topography that could be leveled and irrigated only at excessive expense and areas with general slopes greater than 10%.

Drainage Conditions.—

Areas to be classed as non-arable on account of drainage alone only in the case of stream bottoms, where it is obvious that the cost of artificial drainage will be excessive, and in depressed areas without outlets, although adverse drainage conditions as a contributing factor along with adverse soil conditions will result in further exclusions.

Detailed Land Classification

In detailed land classification the arable land is typically separated into three classes:

(1) Class 1 includes lands of relatively high utility for irrigation farming and suited to a wide range of crops.

(2) Lands in class 2 are suitable for irrigation farming but are less desirable than those in class 1 either because they will produce less or will be more expensive to farm under irrigation. They may have a lower water-holding capacity, as indicated by light texture or limited soil depth; they may be only slowly permeable to water because of clay layers or slightly penetrable hardpan in the subsoil; or they may be more expensive to irrigate because of steeper slopes, or more expensive to level because of uneven topography. Also, they may be restricted in soil depth, moderately saline, or have less favorable drainage characteristics than the class 1 lands.

(3) Class 3 is made up of land that is likewise suitable for irrigation development but is of restricted utility because of more extreme deficiency in one or more respects. Land placed in this class because of relatively steep slopes, but with first-class soil, may be very desirable for fruit production because of favorable air drainage and can be safely irrigated if a permanent cover crop is

maintained between the trees to prevent washing. On the other hand, class 3 land that is so classed because of inferior soil will be restricted to a smaller percentage of row crops in the rotation or, if of limited depth to impenetrable materials, may need to be kept in crops that shade the ground in order to limit saline accumulations. Class 3, therefore, includes lands on which the hazards of farming and also of collecting construction assessments are more acute than in the case of the higher classes.

The land classification of the Columbia Basin project in Washington, which was completed in the summer of 1941, is a good example of the "Detailed Land Classification." The order of land studies on this project included: First, the retracement of land lines; second, topographic surveys, in which maps were prepared having a contour interval of 2 ft and a horizontal scale of 400 ft to the inch; third, the land classification; and fourth, land appraisal. The results of each phase of the investigation were utilized fully in each successive phase. The topographic maps were used as base maps for the land classification, section sheets being prepared showing the various features of the previous surveys, with spaces provided for recording the results of soil profile examinations, analytical data, and notes. The physical standards that were developed on the basis of actual experience, as described in an earlier paragraph, are listed in Table 1. Although these standards were set up as a general guide, they were adhered to closely, except in cases where the peculiar combination of conditions required slight deviation. The limitations of soil depth, and the like, represent minimum requirements for the class, the average being considerably better than these minimum limits. For example, although the permissible minimum limit of 12 in. of loam above gravel or coarse sand was permitted for class 3, this minimum depth was approached in this class only in the case of small-sized spots in areas of larger size where the average depth was considerably greater, and where the topography was such that little or no leveling of the more shallow phases would be required. Lands classed as nonarable because of limited depth, on the other hand, will include many small-sized bodies of land that will exceed this minimum permissible depth for class 3. In other words, it is impossible to set up a set of classification standards that will cover all combinations of conditions encountered, and the individual land classifier was expected to use his judgment in following the "standards," to the end that class 1 would include the land of highest productive capacity with no specific limitations in utility; class 2 also would include good average arable land with a wide range of utility; and class 3, although of more restricted capability, still would be land that could be utilized successfully under irrigation if handled with reasonable care.

Table 2 contains a hypothetical map which, with Fig. 1, illustrates the principal types of soil and topographic conditions encountered on the Columbia Basin project and the manner in which they were handled in the land classification. Although this example is hypothetical in the sense that no single square mile exists on the project which is the counterpart of this map, it was made up by combining parts of several maps and therefore represents actual field conditions. No attempt has been made to have the acreage in the different

TABLE 1.—LAND CLASSIFICATION STANDARDS.—COLUMBIA BASIN PROJECT

Land characteristics	Class 1—Arable	Class 2—Arable	Class 3—Arable	Class 6—Non-Arable
(a) SOILS				
Texture	Sandy loam to friable clay loam.	May be loamy sand if underlain by finer textured soil.	Loamy sand to friable clay.	Includes lands which do not meet the minimum requirements for Class 3. Also small areas of arable land lying within larger bodies of non-arable land when these would obviously not make usable fields. Arable areas which are isolated for irrigation by gravity (high) or those where the feasibility of making water delivery is questionable will be classified in the regular way and the matter of eliminating arable land because of cost of serving with irrigation water will be left for future determination.
Depth:				
To gravel, sand or cobble	30 in. plus.	20 in. plus.[a]	12 in. plus.[a]	
To solid rock or impervious hardpan	48 in. plus.	36 in. plus.	36 in. plus, where flat or hummocky; 24 in. on smooth slopes.	
To caliche hardpan (slightly penetrable)	30 in. with 48 in. penetrable.	24 in. with 36 in. penetrable.	16 in. with 24–36 in. penetrable.	
To penetrable lime zone	18 in. with 48 in. penetrable.	14 in. with 36 in. penetrable.	8 in. with 24–36 in. penetrable.	
Alkali:				
Black alkali	pH under 9.0 in surface 3 ft; may have higher pH in subsoil if highly calcareous and total salts are low.	pH under 9.0 in surface 2 ft; may have higher pH below 2 ft if highly calcareous and total salts are low.	pH under 9.0 in surface 12 in.; may have higher pH below 12 in. if highly calcareous and total salts are low.	
Total salts	0.2 of 1% or less.	0.30 of 1% or less.	0.5 of 1% or less.	
Rock	None in the plow zone of a size that will interfere with cultivation.	Only scattering loose rock large enough to interfere with cultivation.	Not more than are generally cleared for cultivation.	
(b) TOPOGRAPHY				
Slopes	Up to 5% if reasonably large sized bodies slope in the same plane.	Up to 10% if reasonably large sized bodies slope in the same plane.	Up to 15% if slopes are very favorable.	
Surface	Smooth enough that leveling can be accomplished almost entirely with a float.	Moderate grading may be required.	May require heavy grading in spots but not more than has been general on similar developed projects.	
(c) DRAINAGE				
Soil and topography	Soil and topographic conditions such that no specific drainage requirement is to be anticipated.	Soil and topographic conditions such that drainage will probably be required but with reclamation by artificial means appearing feasible at reasonable cost.	Narrow valleys, etc., where drainage will probably be required. Also areas requiring surface outlets where these can be provided at reasonable cost.	

[a] At the minimum depths the soil should be sandy loam or heavier. Loamy sands underlain by coarse sand or gravel should be 20 to 30 in. deep (depending on the percentage of silt and clay) to qualify for class 3. Although areas may be placed in a lower class than "1" for any of the reasons listed above, they will generally be placed in the next lower class if decidedly inferior in two or more respects. The reasons for placing areas in a lower class than "1" should be indicated by placing the letters "S," "T," or "D" after the classification number, depending on whether the deficiency is in "soil," "topography," or "drainage." The boundary between the different classes should be delineated with a solid line and the boundary between areas placed in the same class for different reasons should be a broken line.

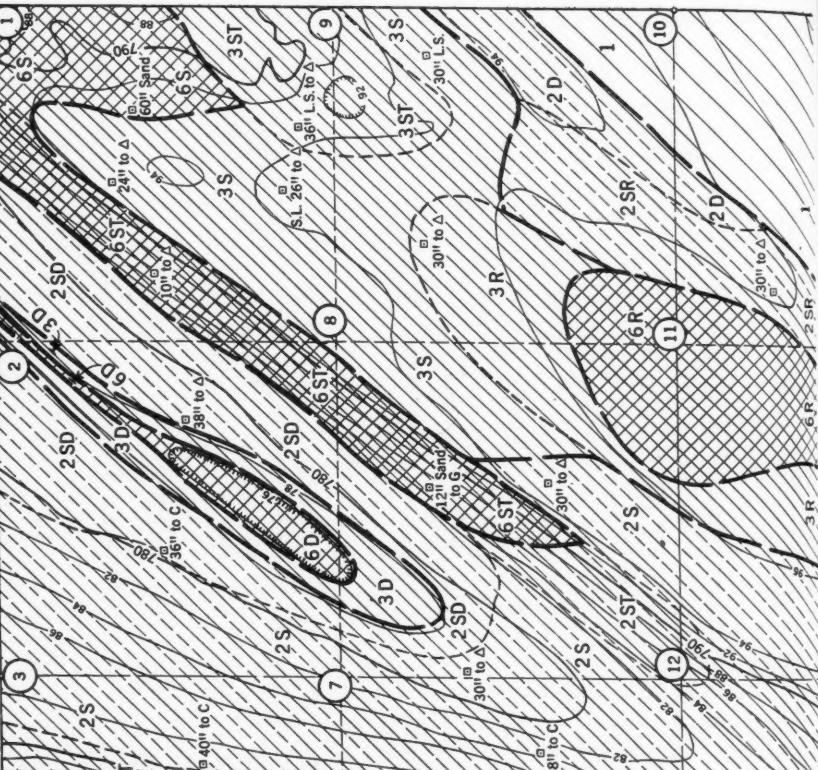
classes on this map proportional to the project areas, the extent of the more inferior types being relatively large on the hypothetical map in order that the maximum number of different conditions encountered in the land classification might be illustrated.

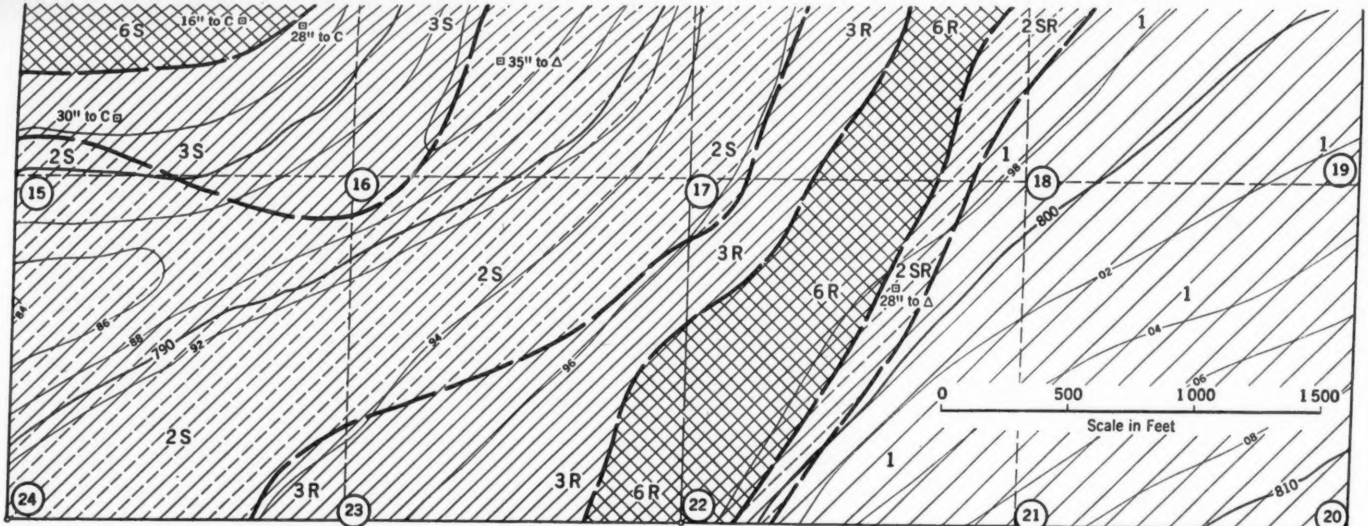
It will be noted that the results of twenty-four soil-profile examinations are recorded in Fig. 1. Conventional symbols, as illustrated in the upper left-hand corner, are used to indicate the soil texture and nature of materials encountered in the 5-ft profile examined. The location of each profile in Fig. 1 is indicated on the map of Table 2 by the number of the profile in a circle located on the map at a point corresponding to the place where the profile examination was made in the field. At twenty-eight other locations the soil depth to various materials is indicated directly on the map as, for instance, in the south center of the NE¼NW¼ where the small "square" appears, representing the location of a place where the depth was indicated as 35 in. to gravel. At such locations less detailed soil-profile examinations are made than at the regularly numbered profiles. These extra examinations are necessary in many cases in order to delineate properly the boundaries between classes.

It will be noted that in the northwest part of the section the land is class 1. Profiles 4 and 5 and the two "spot holes" near the center of the northwest quarter section (Fig. 1) show that this is a deep soil of medium texture, and the topography, although perhaps not ideal, is very good for irrigation purposes. Also, there is no indication of future drainage troubles. In the vicinity of profile 6, the classification is marked "2T," which indicates that, although the soil is good enough for class 1, the land has been placed in class 2 because the slope is somewhat steeper and the direction of fall is not uniform. The strip of land below profile 6, marked "6T," is in excess of 15% in general gradient and is being classified as non-arable on this account, in spite of the fact that the soil is of high quality. The classification of the more moderate slope below is "3T" and where the slope is still more favorable it changes to "2T." Farther south, in the W½ of the SW¼, the topography and drainage conditions are sufficiently favorable for classification as class 1, but there are soil deficiencies, as indicated by the letter "S" following the classification numbers. The classification in the NW¼SW¼ is class 3 and class 6 because of limited depth of soil above caliche (a slightly penetrable calcareous hardpan). The land in class 6 is too shallow for safe utility as irrigated land, and the land in class 3 is deep enough to be arable but has distinct limitations because of shallow soil depth.

Crossing the shallow "draw" and divide, in the S½ of the SW¼, a low terrace is encountered that is underlain by clear sand and gravel. This is classed as "2S" because of the limited depth of good soil above this loose material, which condition is indicated by profiles 17 and 24, Fig. 1. The areas to the east marked class 3R and class 6R, respectively, are made up of soils as desirable as the 2S area except for the presence of cobble and boulders in the plow zone, which has resulted in the least seriously affected part being classed as arable (class 3) and the remainder, where the boulders are too large and too thick to be removed economically to permit cultivation, being classed as non-arable (class 6). The narrow strip of "2SR" was so classed both because of limited

TABLE 2.—HYPOTHETICAL EXAMPLE, LAND CLASSIFICATION; AREAS IN ACRES (SEE FIG. 1)





Class	Southeast Quarter Section				Southwest Quarter Section				Northwest Quarter Section				Northeast Quarter Section				Total
40's	SE	SW	NW	NE	SE	SW	NW	NE	SE	SW	NW	NE	SE	SW	NW	NE	
1	40.0	15.6	0.9	31.7	.....	.....	.....	.....	2.2	14.2	37.0	16.2	3.8	.....	.....	.....	161.6
2		5.9	16.1	6.4	20.1	36.6	3.2	25.1	28.0	11.0	3.0	15.1	11.9	21.2	30.5	6.4	240.5
3		3.1	9.6	.....	16.0	3.4	17.2	13.8	4.7	9.9	.....	2.1	21.5	10.2	4.9	19.2	135.6
6		15.4	13.4	1.9	3.9	.....	19.6	1.1	5.1	4.9	.....	6.6	2.8	8.6	4.6	14.4	102.3
1–3	(40.0)	(24.6)	(26.6)	(38.1)	(36.1)	(40.0)	(20.4)	(38.9)	(34.9)	(35.1)	(40.0)	(33.4)	(37.2)	(31.4)	(35.4)	(25.6)	(537.7)
Total	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	640.0

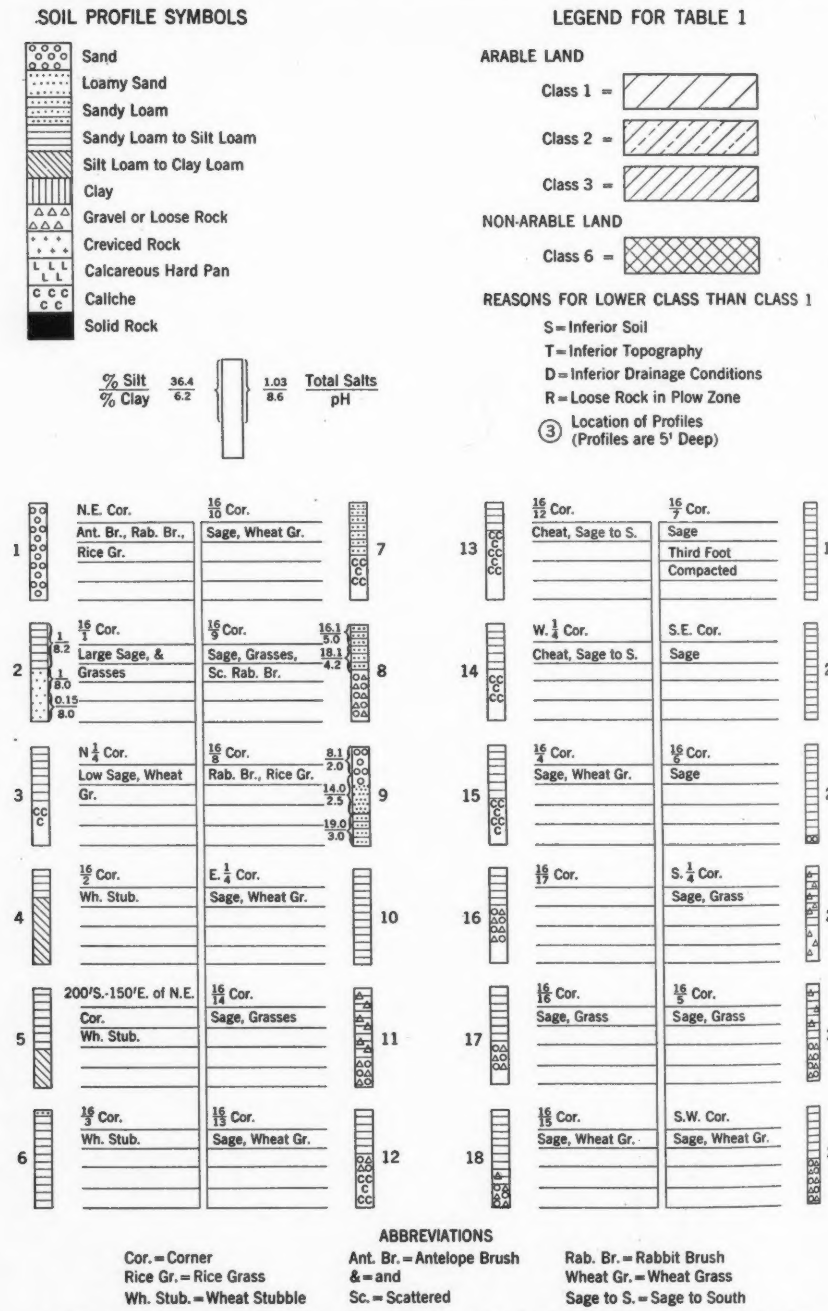


FIG. 1.—HYPOTHETICAL PROFILES, ILLUSTRATING CLASSES AND SUB-CLASSES FOR PRINCIPAL SOIL AND TOPOGRAPHIC CONDITIONS OF PROJECT (SEE TABLE 2)

soil depth above gravel and because of some boulders in the plow zone which will need to be removed to permit cultivation. An area of very fine soil and excellent topography occurs in the southeastern part of the section, which is in class 1.

Another set of conditions is encountered in the N½ of the NE¼. The class 6S area in the eastern part has topography that would permit irrigation, but this land is classed as non-arable because the soil is a loose sand. The area just to the south, which is marked "3ST," is in the lowest arable class for both "soil" and "topographic" reasons, the soil being sand and loamy sand, with a sandy loam subsoil, and the surface being rough enough to require a rather large amount of leveling. The topography in this area differs from that in the "3T" area, previously described, by having a rough surface of moderate gradient, whereas the first area was smooth but steep. The area of class 3S to the west differs from the class 2 land which it joins on the south only in being of more limited depth above gravel and sand. The rather steep slope marked "6ST" is non-arable both because of shallow soil and unfavorable topography.

The area immediately to the west of the last-described land illustrates the classification of three types of drainage conditions. This entire area meets the soil specifications for class 2. It is anticipated, however, that seepage will develop after the adjoining higher lands are placed under irrigation, which will be more serious in the class 6 and class 3 areas. It certainly would not be economical to build a drainage ditch unless it is necessary for the class 2 area, and in this case it undoubtedly would not be deep enough to protect the class 6 land, and the class 3 land probably would have to be used for pasture. The area of class 2S at the N¼ corner has a slope that would facilitate drainage and that is sufficiently smooth and low enough in gradient to be placed in class 1, but it is placed in class 2 because of the limited depth above caliche.

The more general types of land classification, as described, provide the basis for the determination of irrigable acreage and, as far as land is concerned, the basis for project planning prior to the construction stage. The more detailed type, as described for the Columbia Basin, provides information necessary for detailed project planning, for the allocation of construction assessments, and for land appraisal as required by federal laws pertaining to irrigation projects. Much use is being made of the land-classification maps on the Columbia Basin project by land purchasers who are rapidly learning that the information is available. The information provided is expected to become increasingly useful as the different blocks are placed under irrigation.

Conclusions

The system of land classification described, including, as it does, the elimination of inferior (non-arable) lands from those to be irrigated and the division of the "arable" lands into classes on the basis of physical characteristics which determine utility for farming under irrigation, is unavoidably subject to errors in judgment. Every effort is made, however, to hold such errors to a minimum by basing the standards or specifications for each class on demonstrated experience on operating irrigation projects.

Much of the strength of the method lies in its singleness of purpose but a method including a larger number of classes would provide a better basis for determining suitable crops and farm practices. The work is made more useful in the latter respects than would be the case otherwise by the inclusion of a record of soil profile examinations on each land classification sheet.

The determination of the proper point of division between "arable" and "non-arable" land is not easy and is subject to much difference of opinion. Obviously the limitation of the irrigable area of an irrigation project to "ideal" land would not be to the public interest unless the water supply were only sufficient for such an area. On the other hand, the inclusion of land in the irrigable area which will not produce sufficiently well to maintain a farm family, results only in loss to the settler and to the construction agency. It appears generally safe to include, in the irrigable area, lands of a type that have remained in continued production on older projects and to exclude lands similar to those on developed projects which have been abandoned after years of trial.

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PAPERS

MOMENT BALANCE: A SELF-CHECKING ANALYSIS OF RIGIDLY JOINTED FRAMES

BY R. J. CORNISH,[1] ESQ.

SYNOPSIS

The paper describes a method of analysis that has two main features: (a) If the end moments have been estimated previously, the estimate can be adjusted rapidly (a preliminary estimate is not essential, however); and (b) the arithmetical work is almost entirely self-checking.

1. NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

2. THEORY

The method depends on an easily established relationship between the end moments in the members meeting at a rigid joint. In Fig. 1, four members of uniform moment of inertia are shown meeting at point O. The stiffness K_{ON} is equal to $\frac{I_{ON}}{L_{ON}}$, etc.

The slope-deflection relationships give rise to the following expressions for such members:

$$6\,E\,K_{ON}\,\phi_O = 2\,M_{ON} - M_{NO} + X_{ON} \qquad (1a)$$
$$6\,E\,K_{OP}\,\phi_O = 2\,M_{OP} - M_{PO} + X_{OP} \qquad (1b)$$
$$6\,E\,K_{OQ}\,\phi_O = 2\,M_{OQ} - M_{QO} + X_{OQ} \qquad (1c)$$

and

$$6\,E\,K_{OR}\,\phi_O = 2\,M_{OR} - M_{RO} + X_{OR} \qquad (1d)$$

in which E = modulus of elasticity; ϕ_O = the slope of the members at point O; M_{ON} = external bending moment at end O of member ON (etc.); and X = a

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by October 1, 1942.

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term that usually appears in the slope-deflection equation as

$$X = \frac{6\,A}{L^2}\,(L - \bar{x}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The subscript *ON* (etc.) denotes, as before, the end O of member ON. When the moment of the transverse load about the joint under consideration is clockwise, X is positive. In Eq. 2, A is the area of a "simply supported" bending moment diagram; $\bar{x}$ is the distance of its center of area from point O; and L = the effective length of the member.

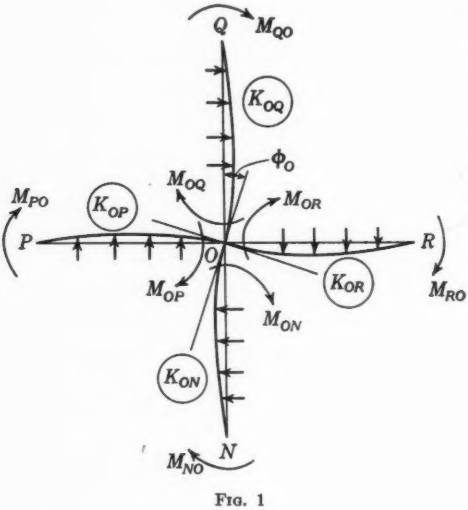


FIG. 1

Addition of Eqs. 1 shows that

$$6\,E\,(\Sigma K)\,\phi_O = \Sigma M' + \Sigma X \ldots(3)$$

since

$$M_{ON} + M_{OP} + M_{OQ} + M_{OR} = 0 \ldots\ldots(4)$$

Then, by substitution for $6\,E\,\phi_O$ from Eq. 3 in Eq. 1*a*,

$$M_{ON} = \frac{1}{2}\left[M_{NO} - X_{ON} + \frac{K_{ON}}{\Sigma K}\,(\Sigma X - \Sigma M')\right] \ldots(5)$$

or, in general,

$$M = \tfrac{1}{2}\,(M' - X + \kappa) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

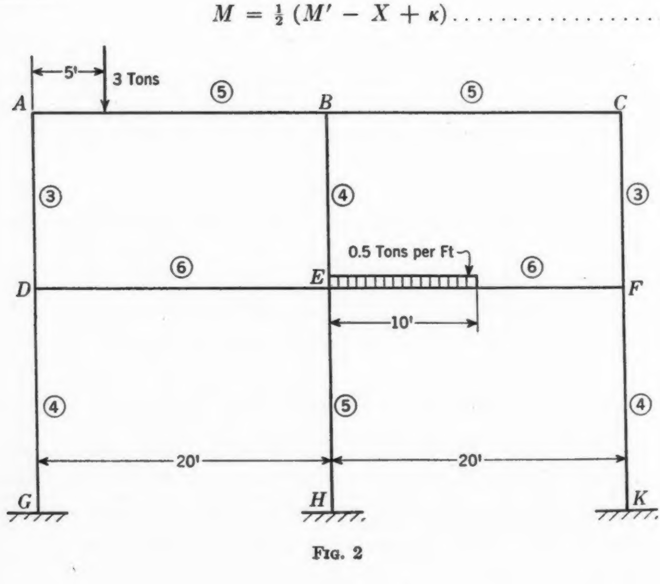


FIG. 2

in which, for simplicity,

$$\kappa = \frac{K}{\Sigma K}(\Sigma X - \Sigma M') \quad \text{.....................(7)}$$

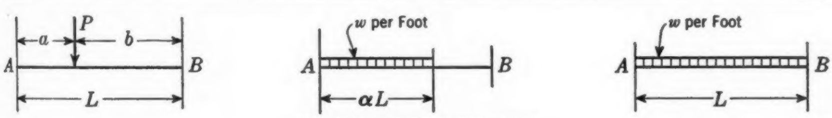
In Eq. 6, M is the external moment acting on the member at O, the joint under consideration, and M' is the moment at the outer end; all external moments are considered positive when clockwise.

The usual assumption is made that the frame can be represented by a line diagram. The prevention of linear displacements means that sidesway and longitudinal strains in the members are neglected, the problem of sidesway being discussed subsequently as a separate refinement.

3. Example

The use of Eq. 6 in frame analysis is best illustrated by an example. Fig. 2 shows a simple frame, carrying a concentrated load of 3 tons on the beam AB and a distributed load of 0.5 ton per ft on the left half of beam EF. The column ends, G, H, and K, are fixed vertically; and the stiffnesses of the various members are shown in the circles.

TABLE 1.—Values of X and the Fixed-End Moment M_F



(1) Single Concentrated Load — UNIFORMLY DISTRIBUTED LOAD: (2) On Part Span; and (3) On Entire Span

No.	A	X_{AB}	X_{BA}	$M_{F(AB)}$	$M_{F(BA)}$
1	$\frac{1}{2} P a b$	$\frac{P a b}{L^2}(L + b)$	$-\frac{P a b}{L^2}(L + a)$	$-\frac{P a b^2}{L^2}$	$\frac{P a^2 b}{L^2}$
2	$\frac{w \alpha^2 L^3}{12} \times (3 - 2\alpha)$	$\frac{w \alpha^2 L^2}{4} \times (4 - 4\alpha + \alpha^2)$	$-\frac{w \alpha^2 L^2}{4} \times (2 - \alpha^2)$	$-\frac{w \alpha^2 L^2}{12} \times (6 - 8\alpha + 3\alpha^2)$	$\frac{w \alpha^3 L^2}{12} \times (4 - 3\alpha)$
3	$\frac{1}{12} w L^3$	$\frac{1}{4} w L^2$	$-\frac{1}{4} w L^2$	$-\frac{1}{12} w L^2$	$\frac{1}{12} w L^2$

Values of X and the Fixed-End Moment M_F.—These values can be obtained at once with the aid of Table 1, and are as follows:

$$X_{AB} = 19.7 \text{ ft-tons}; \quad X_{BA} = -14.1 \text{ ft-tons}; \quad X_{EF} = 28.1 \text{ ft-tons};$$
and
$$X_{FE} = -21.9 \text{ ft-tons}.$$
$$M_{F(AB)} = -8.4 \text{ ft-tons}; \quad M_{F(BA)} = 2.8 \text{ ft-tons}; \quad M_{F(EF)} = -11.5 \text{ ft-tons};$$
and
$$M_{F(FE)} = 5.2 \text{ ft-tons}.$$

Values of X_{AB} (Table 1) are used when balancing joint A, and X_{BA} when

balancing joint B. As a check on the foregoing, note that, in Table 1,

$$X_{AB} - X_{BA} = \frac{6\,A}{L} \quad \text{(8a)}$$

and

$$M_{F(BA)} - M_{F(AB)} = \frac{2\,A}{L} \quad \text{(8b)}$$

Moment Balance.—The next step is to estimate the end moments at the joints. A good designer, guided by the calculated fixed-end moments, can make a very close estimate, and in Fig. 3 the initial estimates have been

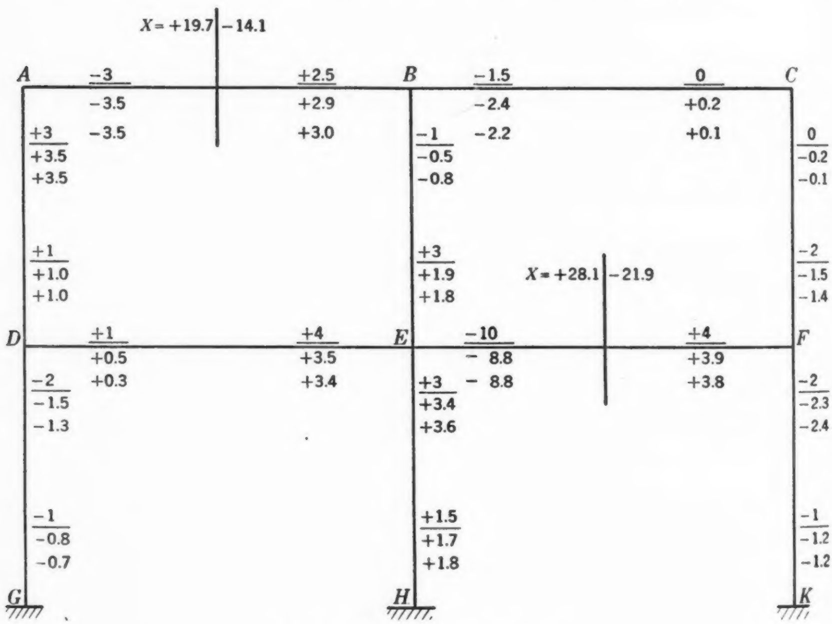


FIG. 3 (Initial Estimates Underlined)

underlined. The moments are balanced in Table 2, successive approximations being entered in Fig. 3 below the initial (underlined) estimates.

In the table there is a column for each member meeting at a joint, and for each joint there is a "Σ" column also. Lines 1 to 5 show the first balance and lines 6 to 10 the second. Joint A has been considered first. In line 1, $-X_{AD}$ is zero, $-X_{AB}$ is -19.7, therefore $-\Sigma X$ is -19.7. The estimated values (+ 1, + 2.5) of M_{DA}, M_{BA} are entered in line 2, giving $\Sigma M' = +3.5$ and $\Sigma X - \Sigma M' = +19.7 - 3.5 = +16.2$. This value is distributed (line 3 and Eq. 7) between the members in proportion to their stiffnesses. (There is no object in computing distribution factors. It is only necessary to set ΣK (= 8) under $\Sigma X - \Sigma M'$ (= 16.2) on the top scale of the slide rule, and

TABLE 2.—Example of Computation Sheet

No.	Quantity	Joint A: Member AD	Joint A: Member AB	Joint A: Σ	Joint B: Member BA	Joint B: Member BE	Joint B: Member BC	Joint B: Σ	Joint C: Member CB	Joint C: Member CF	Joint C: Σ
..	K	3	5	8	5	4	5	14	5	3	8
					(a) First Balance						
1	$-X$	0	−19.7	−19.7	+14.1	0	0	+14.1	0	0	0
2	M'	+1	+ 2.5	+ 3.5	− 3.5	+3	0	− 0.5	−2.4	−2	−4.4
3	κ	+6.1	+10.1	+16.2	− 4.8	−4.0	−4.8	−13.6	+2.8	+1.6	+4.4
4	2 M	+7.1	− 7.1	....	+ 5.8	−1.0	−4.8	....	+0.4	−0.4	....
5	M	+3.5	− 3.5	....	+ 2.9	−0.5	−2.4	....	+0.2	−0.2	....
					(b) Second Balance						
6	$-X$	0	−19.7	−19.7	+14.1	0	0	+14.1	0	0	0
7	M'	+1.0	+ 2.9	+ 3.9	− 3.5	+1.9	+0.2	− 1.4	−2.2	−1.5	−3.7
8	κ	+6.0	+ 9.8	+15.8	− 4.6	−3.6	−4.6	−12.7	+2.3	+1.4	+3.7
9	2 M	+7.0	− 7.0	....	+ 6.0	−1.7	−4.4	....	+0.1	−0.1	....
10	M	+3.5	− 3.5	....	+ 3.0	−0.8	−2.2	....	+0.1	−0.1	....

TABLE 2.—(*Continued*)

No.	Quantity	Joint D: Member DA	Joint D: Member DG	Joint D: Member DE	Joint D: Σ	Joint E: Member ED	Joint E: Member EB	Joint E: Member EH	Joint E: Member EF	Joint E: Σ	Joint F: Member FE	Joint F: Member FC	Joint F: Member FK	Joint F: Σ
..	K	3	4	6	13	6	4	5	6	21	6	3	4	13
						(a) First Balance								
1	$-X$	0	0	0	0	0	0	0	−28.1	−28.1	+21.9	0	0	+21.9
2	M'	+3.5	−1	+4	+6.5	+0.5	−0.5	+1.5	+ 4	+ 5.5	− 8.8	−0.2	−1	−10.0
3	κ	−1.5	−2.0	−3.0	−6.5	+6.5	+4.3	+5.3	+ 6.5	+22.6	− 5.5	−2.8	−3.6	−11.9
4	2 M	+2.0	−3.0	+1.0	....	+7.0	+3.8	+6.8	−17.6	....	+ 7.8	−3.0	−4.6	....
5	M	+1.0	−1.5	+0.5	....	+3.5	+1.9	+3.4	− 8.8	....	+ 3.9	−1.5	−2.3	....
						(b) Second Balance								
6	$-X$	0	0	0	0	0	0	0	−28.1	−28.1	+21.9	0	0	+21.9
7	M'	+3.5	−0.8	+3.5	+6.2	+0.3	−0.8	+1.7	+ 3.9	+ 5.1	− 8.8	−0.1	−1.2	−10.1
8	κ	−1.4	−1.9	−2.9	−6.2	+6.6	+4.4	+5.5	+ 6.6	+23.0	− 5.5	−2.7	−3.6	−11.8
9	2 M	+2.1	−2.7	+0.6	....	+6.9	+3.6	+7.2	−17.6	....	+ 7.6	−2.8	−4.8	....
10	M	+1.0	−1.3	+0.3	....	+3.4	+1.8	+3.6	− 8.8	....	+ 3.8	−1.4	−2.4	....

the distributed moments are read over $K = 3$ and $K = 5$.) Lines 1, 2, and 3 are added to give 2 M (line 4), and the values of M in line 5 are entered in Fig. 3. This completes the first balance of joint A.

For the first balance of joint B, the revised estimate − 3.5 is used for M_{AB}; M_{EB} and M_{CB} have been estimated at + 3 and 0, respectively; and the work

proceeds as before. Joints C, D, E, and F are balanced in turn. Since DG, EH, and FK are direction-fixed at G, H, and K, the estimates of M_{GD}, M_{HE}, and M_{KF} are revised by dividing each fresh estimate of M_{DG}, M_{EH}, and M_{FK} by two.

As the original estimates were fairly close to the actual, the first balance gave results precise enough for practical purposes. However, a second balance of all the joints has been made in this example.

A few points may be noted:

(1) The accuracy of the original estimate does not affect the accuracy of the result. In the example, if the moments at all joints are initially assumed to be zero, results of the same order as those of Fig. 3 can be obtained with two balances.

(2) The fact that, to a large extent, the work is self-checking is evident. It is easy to check whether the final values of M (lines 5 and 10) at each joint add up to zero, and even if an error is made it is eliminated in subsequent balances.

(3) Although, for convenience of demonstration, the joints were balanced in the order A, B, C, D, E, F, it is probably best to start the first balance at E, and proceed in the order E, F, A, B, C, D. For the second balance some other order, which can be quickly decided by inspection, may be better.

(4) If $-\frac{1}{2}X$ and $\frac{1}{2}M'$ are used in lines 1 and 2 of Table 2 instead of $-X$ and M', line 3 will also be half as great, and the total of lines 1, 2, and 3 will be M instead of $2M$. Thus the written work for each balance can be reduced by 20% at the expense of a possible source of inaccuracy, in that one must remember each time to divide the values of M', shown in Fig. 3, by 2.

4. Summary of Procedure

The basic procedure of the Method of Moment Balance may be summarized briefly as follows:

(*a*) The term X and the fixed-end moment M_F are computed for all members carrying transverse loads;

(*b*) Reasonable values are assumed for the end moments and are entered on the design sheet;

(*c*) The joints are balanced successively with the aid of Eq. 6, beginning at a member carrying a transverse load;

(*d*) When one, or more, of the members connected to a joint can be regarded as direction-fixed at the outer end, the moment must be carried over to that end. If, in Fig. 1, ON is fixed vertically at N,

$$M_{NO} = \frac{1}{2}(M_{ON} - X_{NO}) \qquad (9)$$

In the case of building frames with vertical loads, X for the columns is zero, and half the moment at the top of each of the lowest columns is carried over to the base.

5. Sideway

With an unsymmetrical system of vertical loads in a structure there is linear displacement of the joints as well as rotation. Except in the case of very simple frames such as the single "portal," the moments due to this displacement can be neglected, since they make very little difference to the moment at any point when all dead and live loads are taken into account.

Sidesway moments of some consequence may be caused by horizontal wind loads. The method of moment balance can be applied to such cases in a manner similar to that already described herein.

Referring to Fig. 1, if the transverse loads are omitted and the points N, P, Q, and R are given deflections Δ_N, Δ_P, Δ_Q, and Δ_R in a clockwise direction, it can be shown easily that, in general,

$$M = \frac{1}{2}\left[M' - \frac{6\,E\,K\,\Delta}{L} + \frac{K}{\Sigma K}\left(6\,E\,\Sigma\,\frac{K\,\Delta}{L} - \Sigma M'\right)\right] \quad \ldots\ldots(10)$$

The analysis then follows the lines suggested by Hardy Cross and N. D. Morgan,[2] Members, Am. Soc. C. E.: "Assume as many different combinations of horizontal displacements as there are degrees of freedom of horizontal movement, and for each combination determine a set of shears corresponding to a given set of moments." This leads to the formation of as many simultaneous equations as there are floors.

The method is rather cumbersome for more than two stories, and for practical purposes another method given by Messrs. Cross and Morgan[3] provides a more rapid analysis of wind-load moments.

6. Acknowledgments

The writer wishes to express his gratitude to Denis Matthews, Jun. Am. Soc. C. E., and G. W. Stokes, for their very helpful criticism and advice in the preparation of this paper.

APPENDIX

Notation

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932:[4]

A = the area of a "simply supported" bending moment diagram due to transverse loads;

[2] "Continuous Frames of Re-inforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 228.

[3] *Ibid.*, pp. 228–233.

[4] ASA—Z10a—1932.

a = distance of a load from the left-hand support;
b = distance of a load from the right-hand support;
E = modulus of elasticity;
I = moment of inertia; second moment of the area of a cross section;
K = stiffness ratio $\frac{I}{L}$: ΣK = sum of the stiffness ratios of members meeting at a joint;
L = effective length of a member, as designated by proper subscripts;
M = external end moment of a member, designated by the proper subscripts (considered positive when clockwise):
  M' = external moment at the outer end of a member (the terms "inner" and "outer" ends are used, respectively, for the ends of a member at, and remote from, the joint under consideration);
  M_F = fixed-end moment;
P = a concentrated load;
w = uniformly distributed load per unit length;
X = a term commonly encountered in the slope deflection analysis (see Eq. 3), a subscript—ON, for example—denoting the end O of member ON; X is positive when the moment of the transverse load about the joint under consideration is clockwise;
$\bar{x}$ = distance of the center of area of a "simply supported" bending moment diagram from the "inner" end of a member (the terms "inner" and "outer" ends are defined under M');
α = proportion of the effective span L covered by a uniformly distributed load w;
Δ = deflection of the "outer" end of a member relative to the "inner" end (the terms "inner" and "outer" ends are defined under M');
κ = an arbitrary substitution factor used to simplify Table 2 (line 3); defined by Eq. 7; and
ϕ = slope of a member at any point.

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PAPERS

NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS

By N. M. Newmark,[1] Assoc. M. Am. Soc. C. E.

Synopsis

A numerical procedure for computing the deflections and moments in beams and columns is described herein. The method is of particular applicability in determining critical buckling loads and configurations of bars of variable cross section loaded in various ways. For such problems the procedure becomes one of successive approximations. By means of a simple modification of the data entailing very little increase in numerical work, considerably greater accuracy is obtainable by this procedure than by others of similar nature hitherto available.

The numerical procedure is approximate, but leads to exact moments (or deflections) when the loading diagram (or diagram of "angle changes") is made up of segments that are bounded by straight lines or by arcs of parabolas. By taking more arbitrary divisions in the length of a bar one obtains more accurate results in the general case. For most practical problems no more than five or six segments are necessary.

The procedure may be applied to other problems which depend on the same general principles. In mathematical terms, the procedure may be applied to the process of numerical integration of certain types of differential equations, in some cases directly, and in other cases by a sequence of successive approximations.

The essential features of the procedure are not new. The writer's first acquaintance with the concepts involved in this paper came some years ago from lectures in graduate courses at the University of Illinois, Urbana, Ill., by Hardy Cross and H. M. Westergaard, Members, Am. Soc. C. E. Specific procedures have been discussed previously in engineering literature; for

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example, the application of similar graphical and analytical procedures to buckling of bars has been made by L. Vianello,[2] F. Engesser,[3] and others[4]; and the application of a graphical procedure to vibration of bars and shafts has been indicated by A. Stodola.[5,6] Methods of obtaining increased accuracy with a numerical procedure have been suggested by A. S. Niles,[7] Assoc. M. Am. Soc. C. E., and R. V. Southwell,[8,9] among others. The procedure suggested by Professor Southwell is in some respects similar to that described herein. However, the generalization of the procedure, the manner of application to specific problems, the treatment of functions with cusps or discontinuities, the simplified procedure for continuous functions with continuous derivatives, and the method of computing shears, slopes, or first derivatives, are essentially new, are more useful or more accurate than previous methods, and to the writer's knowledge have not been described previously.

Part I.—Computation of Moments in Beams

Introductory.—The calculation of the values of a function of a single variable, when the magnitude of the second derivative of the function is known, is a fundamental part of a group of physical problems, examples of which are the determination of the deflection of a string, or of a beam, and the computation of moments in a beam due to given loads. Analogies may be drawn between these and similar problems, since generally they may all be solved by the same procedures.

The method of computation described herein is a numerical procedure permitting as accurate a determination as is desired of the values of a function for specific values of the variable. The method is described in terms of calculation of moments in a beam for a given system of loads, but the application to other problems is also indicated, and particular application is made to the problem of buckling of bars.

Treatment of Concentrated Loads.—A fundamental part of the procedure depends on the rapid and systematic calculation of shears and moments in a beam subjected to a series of concentrated loads. Essentially, the process is to compute the shears from one end of the beam to the other by adding or sub-

[2] "Graphische Untersuchung der Knickfestigkeit gerader Stäbe," by L. Vianello, *Zeitschrift des Vereins Deutscher Ingenieure*, Vol. 42, 1898, pp. 1436–1443.

[3] "Über die Knickfestigkeit von Stäben veränderlichen Trägheitsmomentes," by F. Engesser, *Zeitschrift des österreichischen Ingenieur- und architektenvereins*, Vol. 61, 1909, pp. 544–548.

[4] A discussion of some of these methods is given in "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 84–88 and 131–133.

[5] "Steam Turbines," by A. Stodola and L. C. Loewenstein, 2d Revised Ed., New York, N. Y., 1906, pp. 185–186.

[6] See also, for example, "Mechanical Vibrations," by J. P. Den Hartog, New York, N. Y., 1934, pp. 174–178.

[7] "Airplane Structures," by A. S. Niles and J. S. Newell, 2d Ed., New York, N. Y., 1938, Vol. I, pp. 133–136, and Vol. II, pp. 126–134.

[8] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 155–181, especially pp. 163–165.

[9] "Relaxation Methods Applied to a Spar of Varying Section, Deflected by Transverse Loading Combined with End Thrust or Tension," by R. J. Atkinson, K. N. E. Bradfield, and R. V. Southwell, *Reports and Memoranda No. 1822*, Aeronautical Research Committee, London, 1937.

tracting the successive loads, then to compute the moments by adding or subtracting the successive shears, multiplied by the length of beam over which the shear acts. The latter step is simpler if all the lengths between points of application of the concentrated loads are the same. However, the general case is not difficult, and the modification of the procedure described herein, to handle the general case, is obvious and will not be discussed.

To avoid confusion, a definite sign convention will be adopted in the work that follows. Moments will be considered positive when producing compression in the upper fibers of the beam. Shears will be considered positive when the resultant force to the left of a section is upward. Loads will be considered positive when the load acts upward. The latter convention is chosen so as to permit successive calculation of shears or moments always by adding, respectively, loads or shears, from left to right, and by subtracting the proper quantities from right to left.

When the shear or moment at any point is known the calculation can always be started from that point, but when only the moments at two points are known, the calculation of shears cannot be performed directly. However, a linear moment diagram, which corresponds to a constant shear, and therefore to no load, can always be added to the moments computed from some arbitrary shear chosen to start the calculations. Therefore, one may obtain the desired conditions relatively simply by merely adding a straight-line moment diagram as a correction, where it is needed.

The procedure is simplified by omitting the multiplication of the shears by the distance between loads until the end of the computations. That is, one can consider the loads as numerical quantities all multiplied by a common factor. The shears will be obtained from the loads, and will contain the same common factor. Then the moments will be computed as numerical quantities all multiplied by a common factor, which is the factor for the loads multiplied by the distance between loads.

The calculations are illustrated by the group of problems shown in Fig. 1. The units in which the loads are measured and the length of the panels are omitted purposely: These may have any values. The beam is divided into six equal segments, and the loads are shown in Fig. 1(*a*). The loading is the same for the different problems, but the manner of support and the method of performing the computations vary in the problems. In Fig. 1(*b*) the beam is cantilevered from the right end. Therefore at the left end the shear is zero and the moment is zero. In Figs. 1(*c*), 1(*d*), and 1(*e*) the beam is simply supported. In Fig. 1(*c*) are given linear correction moments which may be added to the moments in Fig. 1(*b*) to satisfy conditions of simple support; that is, zero moment at the two ends of the beam. The same result is obtained in Fig. 1(*d*), starting with the loads but choosing the shears so as to obtain the correct moments directly. In Fig. 1(*e*) the procedure is carried through in what might be a more usual calculation. One starts with a shear of five at the left end, merely as a guess. Then a proper correction to the moments is written in. The details of the calculations are self-explanatory.

Treatment of Distributed Loads.—When distributed loads are applied to the beam, one can choose equivalent concentrated loads that produce the same

shears and moments at certain specified sections of the beam, and thereby handle the problem with the aforementioned single procedure. In so far as statics is concerned, the beam with the distributed load applied directly is equivalent to a system of simply supported stringers resting on the beam, and transmitting the distributed load to the beam as a series of concentrations which are the stringer reactions. The statical equivalence is illustrated in

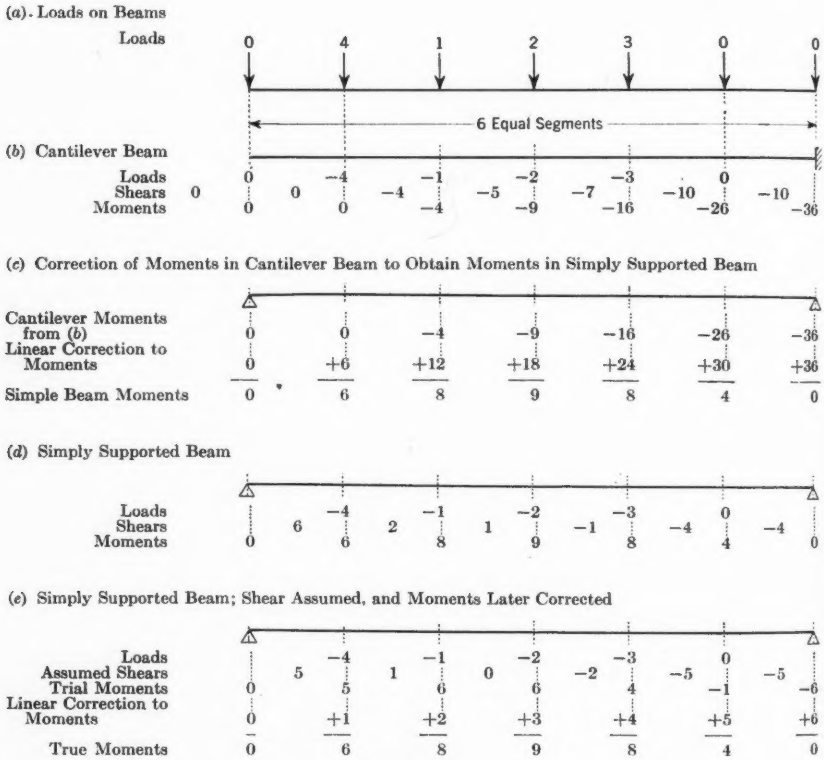


FIG. 1

Fig. 2. It may be observed that the moment or shear at any section of the original beam is equal to the moment or shear at any section through the beam and stringer of the beam-stringer substitute.

One obtains correct moments and, with some care in separating the two subreactions that make up the substitute concentrated load at a point, one obtains correct shears in the original beam at the points of support of the fictitious stringers, by considering a substitute structure loaded only by concentrated loads which are the reactions on the fictitious stringers. One also obtains correct reactions at the ends of the beam.

For a load diagram which consists of straight-line segments, the equivalent concentrated loads are readily determined directly. Formulas for the equiva-

lent loads[10] are stated in Fig. 3. To illustrate the use of the procedure for such a load diagram, several simple problems are shown in Fig. 4. In Fig. 4(*a*), a uniform load on a simply supported beam is considered. Solutions are given in Fig. 4(*b*) for a triangular load diagram on a cantilever beam, and in Fig. 4(*c*)

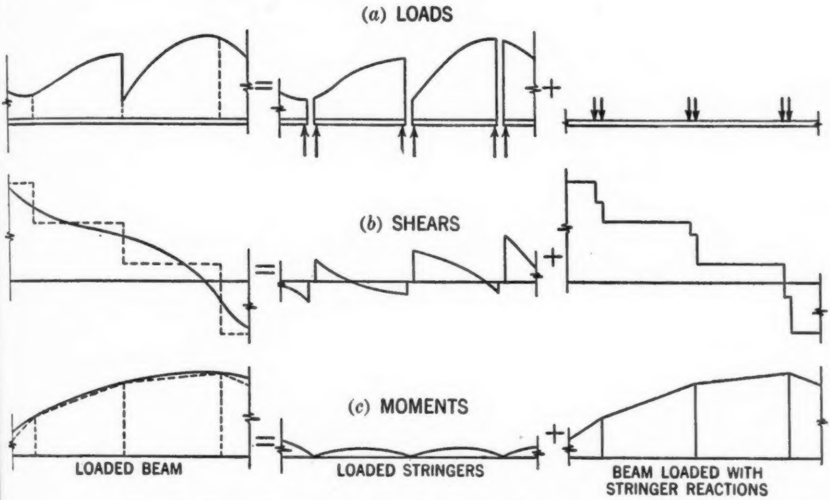


FIG. 2.—STATIC EQUIVALENCE OF BEAM WITH BEAM AND STRINGERS

for a triangular load diagram on a simple beam. In Figs. 4(*a*) and 4(*b*) the shears are computed at intermediate points; consequently the equivalent concentrations are shown in two parts. In Fig. 4(*c*) only the shears at the supports (that is, the reactions) are computed, and therefore only the total equivalent

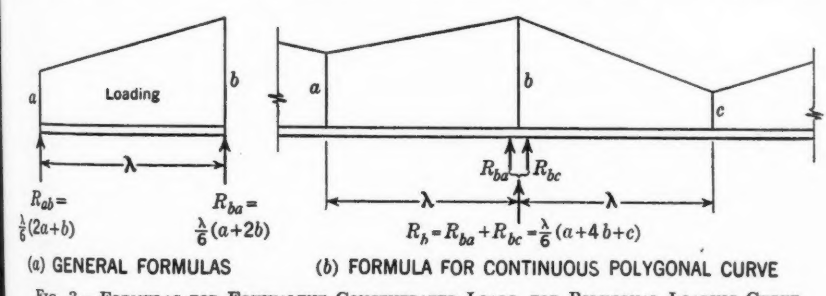


FIG. 3.—FORMULAS FOR EQUIVALENT CONCENTRATED LOADS, FOR POLYGONAL LOADING CURVE

loads are shown. Note that the moments in Fig. 4(*c*) could have been obtained from Fig. 4(*b*) by adding a linear moment diagram.

One can obtain formulas for more complicated types of load distribution. For practical purposes a distribution that varies according to the ordinates to

[10] The same formula for an equivalent concentration for a polygonal loading curve has been given in "Die graphische Statik der Baukonstruktionen," by H. Müller-Breslau, Vol. 2, Pt. II, 2d Ed., Leipzig, 1925, p. 44.

an arc of a second-degree parabola is sufficiently general, since it is possible to approximate almost any curve by segments of second-degree parabolas. Formulas for the equivalent concentrations for such a load are given in Fig. 5,

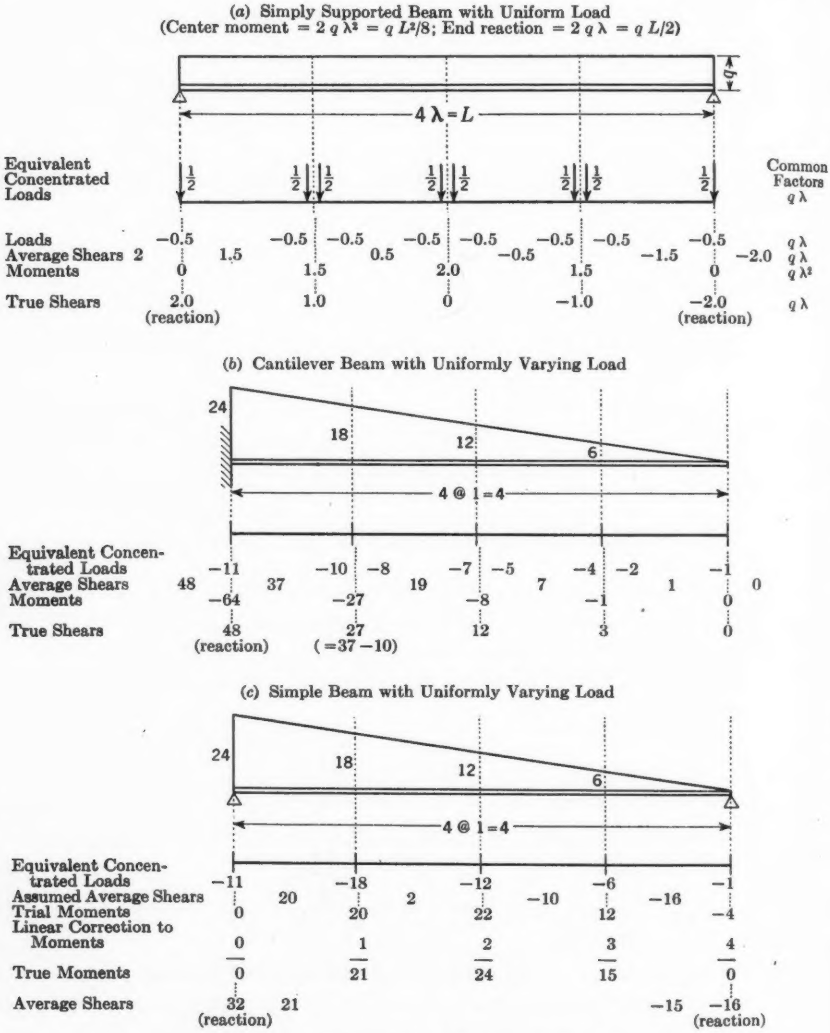


FIG. 4.—APPLICATION OF EQUIVALENT LOADS

in terms of three ordinates to the load distribution curve. The formula in Fig. 5(*b*) for a smooth loading curve was presented by A. Nádai in 1925[11]; and, what amounts to an equivalent formula for the smooth loading curve, with

[11] "Die elastischen Platten," by A. Nádai, Berlin, 1925, p. 209, Eq. 13.

additional terms, was derived by Professor Southwell in 1937.[8] A derivation of the formulas in Fig. 5 appears in the Appendix of this paper. It is noted that one of the three ordinates in Fig. 5(a) need not be an actual ordinate to the loading curve, but can be an extrapolated value. The formulas in Fig. 5 reduce to those in Fig. 3 when $a + c = 2\,b$, or when the parabola becomes a straight line. In general, the formulas in Fig. 5 may be used for any distributed loading to give a reasonably good approximation, and they are used in the remainder of this work.

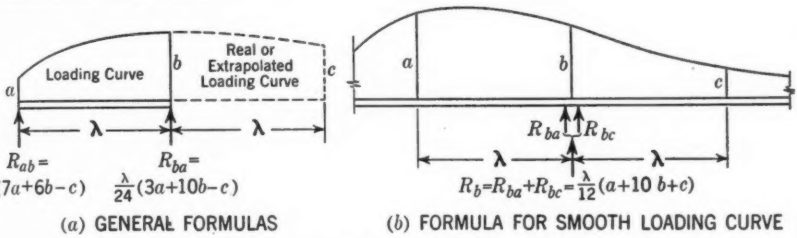


FIG. 5.—FORMULAS FOR EQUIVALENT CONCENTRATED LOADS

Problems in which use is made of the formulas in Fig. 5 are given in Part II. A simpler manner of using the results in Fig. 5 for curves that have no discontinuities, nor abrupt changes in slope, is also illustrated in the material that follows.

PART II.—CALCULATION OF DEFLECTION OF BEAMS

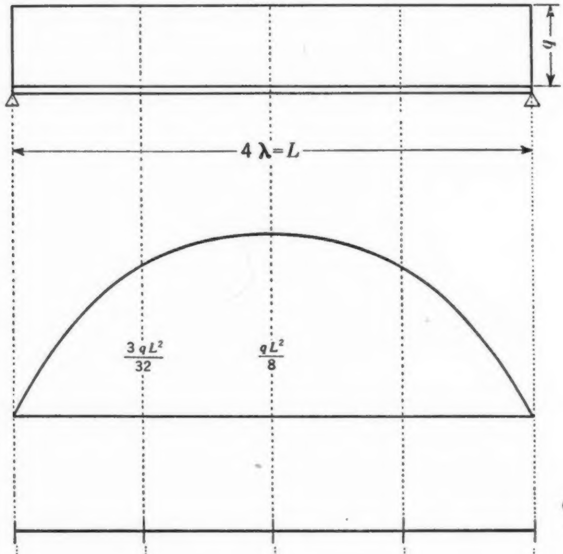
General Relations and Definitions.—A direct analogy can be drawn relating loads, shears, and moments in a beam to "angle changes," slopes, and deflections of a beam,[12] where the "angle changes" are the quantities giving the change of slope per unit length—that is, values of moment M divided by modulus of elasticity E and by moment of inertia I for an elastic beam with small deflections. The following sign convention is adopted in order that the analogy may hold without change of signs.

The "angle change" is defined as $-\frac{M}{E\,I}$; a positive "angle change" is considered as an upward load and therefore as a positive load. Then positive slope corresponds to an increase in deflection from left to right, and corresponds to a positive shear. Finally, positive deflection is taken as downward, and corresponds to a positive moment. A "concentrated angle change" corresponds to an abrupt change in slope at a point, and may be considered in the calculations without difficulty.

As a simple example of the use of the procedure, consider the deflection of a simply-supported beam of constant cross section subjected to uniform load, as in Fig. 6. The moment diagram is a parabola. Therefore the procedure will yield exact results with as many or as few segments as are desired. The calculations are shown for four segments in the length of the beam. The correct center deflection would have been obtained even if only two segments had been

[12] See, for example, "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 28–30.

considered. Note that the constant factors in moment, angle changes, slopes, and deflections are written as multipliers at the right of the calculations. The equivalent concentrated angle changes at the ends of the beam need not be computed if only the deflections are desired.



										Common Factors
Ordinates to Moment Diagrams	0		3		4		3		0	$\frac{q L^2}{32}$
Ordinates to Angle-Change Diagrams	0		−3		−4		−3		0	$\frac{q L^2}{32 E I}$
Equivalent Concentrated Angle Changes	−7		−34		−46		−34		−7	$\frac{q L^2}{32 E I} \frac{\lambda}{12}$
Average Slope		57		23		−23		−57		$\frac{q L^2}{32 E I} \frac{\lambda}{12}$
Deflection	0		57		80		57		0	$\frac{q L^2}{32 E I} \frac{\lambda^2}{12}$

$$\text{Center Deflection} = \frac{80\, q\, L^2}{32\, E\, I} \frac{\lambda^2}{12} = \frac{5\, q\, L^4}{384\, E\, I}$$

$$\text{Quarter Point Deflection} = \frac{57}{80}\ \text{Center Deflection}$$

$$\text{Slope at End of Beam} = (57 + 7) \frac{q\, L^2}{32\, E\, I} \frac{\lambda}{12} = \frac{q\, L^3}{24\, E\, I}$$

FIG. 6.—DEFLECTIONS FOR SIMPLY SUPPORTED BEAM WITH UNIFORM LOAD

Simplified Procedure for Smooth Angle Change Curves.—It can be shown that, for the determination of deflections (or moments) alone, a simpler procedure may be used which avoids the calculation of the equivalent concentrated angle changes (or loads) from a distribution of angle changes (or loads) that has no discontinuities nor abrupt changes in slope in the region considered.

From the formula in Fig. 5(*b*) applying to a smooth curve, one has the relation:

$$R_b = \lambda\, b + \frac{\lambda}{12} (a - 2\, b + c) \qquad (1)$$

Then one can consider the equivalent concentration at any point such as b as made up of two parts: (1) The ordinate to the curve at the point multiplied by λ; and (2) a correction which is $\frac{\lambda}{12}(a - 2b + c)$. The correction loads at all the points, however, produce a deflection that is proportional to the original angle-change curve; actually, deflections at the various points due to the correction are $\frac{\lambda^2}{12}$ times the value of the distributed angle change at the point, plus any linear diagram required to satisfy the boundary conditions. This is obvious from the form of the equation. A proof is demonstrated in Fig. 7.

										Common Factors
		λ		λ		λ		λ		
Ordinates to the Angle-Change Diagram	a		b		c		d		e	
Part (2) of the Equivalent Concentrated Angle Change			$c - 2b + a$		$d - 2c + b$		$e - 2d + c$			$\lambda/12$
Assumed Average Slopes, Part (2)		$b - a$		$c - b$		$d - c$		$e - d$		$\lambda/12$
Deflections, Part (2), with Which Any Lineal Set of Deflections May Be Combined	a		b		c		d		e	$\lambda^2/12$

FIG. 7.—DERIVATION OF PART (2) OF THE DEFLECTIONS FOR A SMOOTH CURVE OF ANGLE CHANGES

The problem of Fig. 6 is solved in Fig. 8 by use of the modified procedure. It is noted that the equivalent concentrated angle changes are not written; consequently the slopes must be computed from the original distributed angle changes multiplied by λ, as indicated in the factors to the right of the calculations. One should be careful that part (2) of the deflection is written with its proper sign. It is always $\frac{\lambda^2}{12}$ times the ordinates to the curve of distributed angle changes, and has the same sign as the distributed angle change.

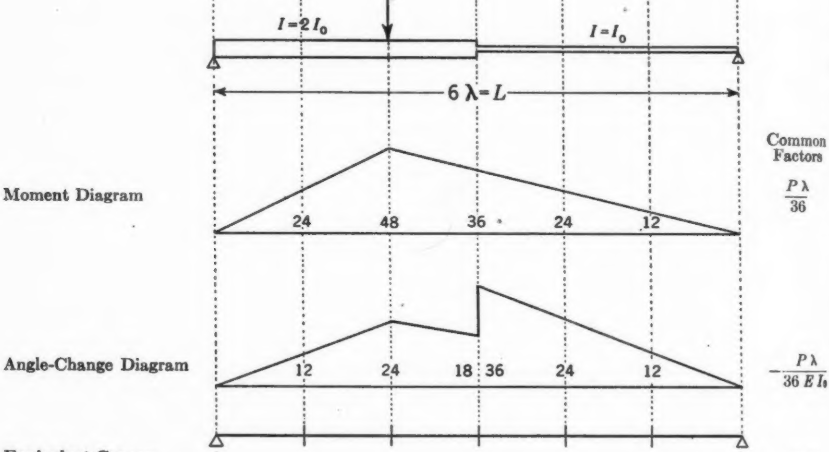
				$4\lambda = L$						
Ordinates to the Angle-Change Diagram	0		−3		−4		−3		0	$\frac{qL^2}{32EI}$
Average Slope, Part (1)		5		2		−2		−5		$\frac{qL^2\lambda}{32EI}$
Deflections, Part (1)	0		5		7		5		0	$\frac{qL^2\lambda^2}{32EI}$
Deflections, Part (2)	0		$-\frac{3}{12}$		$-\frac{4}{12}$		$-\frac{3}{12}$		0	$\frac{qL^2\lambda^2}{32EI}$
Total Deflection	0		$\frac{57}{12}$		$\frac{80}{12}$		$\frac{57}{12}$		0	$\frac{qL^2\lambda^2}{32EI}$

FIG. 8.—ALTERNATIVE PROCEDURE FOR PROBLEM OF FIG. 6

When the original angle-change diagram is linear (that is, either constant or uniformly varying) it is unnecessary to consider part (2) of the deflections since one may add a linear set of deflections to make the net effect of $\frac{\lambda^2}{12}$ times the original angle changes and the added linear deflection zero. Then one may add whatever other linear deflections are required to satisfy the conditions of the problem.

Where there is a break in the curve or a discontinuity in slope, it is impossible to use this simplified procedure without modification. To avoid confusion the general procedure is recommended for such problems.

Further examples of the use of the general procedure and of the modified procedure are given in Figs. 9 and 10, which illustrate respectively the calcula-



														Common Factors
Equivalent Concentrated Angle Changes			−12		−21		−26		−24		−12			$\frac{P\lambda^2}{36 E I_0}$
Assumed Average Slopes		33		21		0		−26		−50		−62		$\frac{P\lambda^2}{36 E I_0}$
Trial Deflections	0		33		54		54		28		−22		−84	$\frac{P\lambda^3}{36 E I_0}$
Linear Correction to Deflections	0		14		28		42		56		70		84	$\frac{P\lambda^3}{36 E I_0}$
True Deflections	0		47		82		96		84		48		0	$\frac{P\lambda^3}{36 E I_0}$

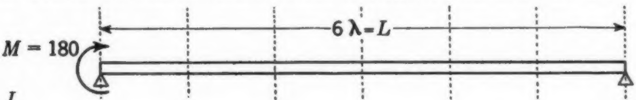
FIG. 9.—DEFLECTION OF BEAM WITH CHANGE IN SECTION

tion of deflections for a member with an abrupt change in section, and for a member of varying cross section. The deflections in Fig. 9 will be exact since the $\frac{M}{E\,I}$-curve is composed of straight-line segments. However, the deflections in Fig. 10 are not exact since the curve of angle changes is not composed of straight-line or parabolic segments. More nearly correct results are obtained by taking more divisions in the length of the beam. The number of divisions actually taken (six) will yield results that are very accurate as is shown by comparison with a solution having twice the number of divisions, in Fig. 10(*b*), and with an "exact" solution in Fig. 10(*c*).

Analyses of Statically Indeterminate Beams.—By superposing the effects of different end moments one can solve the problem of a statically indeterminate beam also. For example, in Fig. 11(*a*) is shown the same beam as in Fig. 10, with a moment applied to the opposite end. The end slopes due to the mo-

ments applied in Figs. 10 and 11 are easily computed and are indicated on the figures. The calculation of end slope from the equivalent concentrated angle change at the end leads to a much greater accuracy than is possible by other means—for example, by methods involving differences of various orders of the final deflections only, as suggested by Professor Southwell.[13]

(a) Deflection of Beam of Variable Section Subjected to Moment at Left End



$\frac{I}{I_0}$	1		2		3		4		5		6		7	Common Factors
Ordinates to Moment Diagram	180		150		120		90		60		30		0	
Ordinates to Angle-Change Diagram	−180		−75		−40		−22.5		−12		−5		0	$1/EI_0$
Assumed Average Slope, Part (1)		100		25		−15		−37.5		−49.5		−54.5		λ/EI_0
Trial Deflections, Part (1)	31.5		131.5		156.5		141.5		104.0		54.5		0	λ^2/EI_0
Deflections, Part (2)	−15.0		−6.3		−3.3		−1.9		−1.0		−0.4		0	λ^2/EI_0
Linear Correction to Deflections	−16.5		−13.8		−11.0		−8.3		−5.5		−2.8		0	λ^2/EI_0
Final Deflection	0		111.4		142.2		131.3		97.5		51.3		0	λ^2/EI_0
End Slopes: From Average Slope	111.4												−51.3	λ/EI_0
From Equivalent Concentrated Angle Change at End	69.6												−0.8	λ/EI_0
Total End Slope	181.0												−52.1	λ/EI_0

(b) Results of Solution with 12 Divisions of Length $\frac{\lambda}{2}$; (values given only for alternate points corresponding to those above)

	1	2	3	4	5	6	7	
Final Deflections	0	110.7	141.4	130.8	97.1	51.1	0	λ^2/EI_0
End Slopes	177.4						−51.9	λ/EI_0

(c) Exact Solution by Integration

	1	2	3	4	5	6	7	
Deflections	0	110.63	141.37	130.76	97.08	51.12	0	λ^2/EI_0
End Slopes	176.75						−51.89	λ/EI_0

FIG. 10.—DEFLECTION OF BEAM OF VARIABLE SECTION BY MODIFIED PROCEDURE

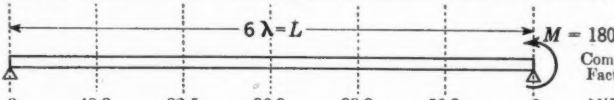
The end slopes in Figs. 10 and 11 differ slightly from the exact values obtained by integration. A much better agreement with the exact values is obtained if a greater number of segments in the length of the beam are used. There is a rapid change in the values of the angle change curve at the left end of the beam in Figs. 10 and 11, and consequently a greater error in the slopes for this end than for the right end, by the approximate procedure. It should also be pointed out that the slope at the right end in Fig. 10(*a*) should be equal to the slope at the left end in Fig. 11(*a*), by Maxwell's theorem of reciprocal deflections. The difference is due to the fact that the procedure involves

[13] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 166–167.

some slight inaccuracies, which amount to analyzing slightly different structures in the two cases.

From the moments and slopes in Figs. 10 and 11, one can find, for example, the stiffness and the carry-over factor for the left end of the beam by adjusting the moments at the ends to give the proper conditions as shown in Fig. 11(c).

(a) Deflection of Beam of Fig. 10 for Moment $M = 180$ at Right End, with Beam Divided into 6 Segments



								Common Factors
Final Deflections	0	48.3	82.5	96.9	88.9	56.9	0	$\lambda^2/E I_0$
End Slopes	51.2						−69.7	$\lambda/E I_0$

(b) Exact Solution by Integration

Deflections	0	48.48	82.66	97.03	88.99	56.98	0	$\lambda^2/E I_0$
End Slopes	51.89						−69.73	$\lambda/E I_0$

(c) Combination of Fig. 10(a) and Fig. 11(a) to Obtain Stiffness and Carry-Over Factor for Left End of Beam

181.0	End slopes, $M = 180$ at left end	−52.1	$\lambda/E I_0$
−38.2	End slopes, $M = -0.747 \times 180$ at right end	+52.1	$\lambda/E I_0$
142.8	Total slope, $M = 180$ at left end, $M = -0.747 \times 180$ at right end	0	$\lambda/E I_0$

Carry-over factor $= -0.747$

$$\text{Stiffness} = \frac{180}{142.8}\,\frac{E I_0}{\lambda} = 1.261\,\frac{E I_0}{\lambda}$$

(d) Exact Values of Carry-Over Factor and Stiffness for Left End of Beam, by Integration

Carry-over factor $= -0.7442$

$$\text{Stiffness} = 1.3031\,\frac{E I_0}{\lambda}$$

FIG. 11.—CALCULATION OF STIFFNESS AND CARRY-OVER FACTOR

For comparison, "exact" values of stiffness and carry-over factor are shown in Fig. 11(d), obtained by integration. The agreement is close although only six segments were used in the approximate procedure.

PART III.—DEFLECTION OF BEAMS WITH AXIAL LOADS; BUCKLING OF COLUMNS

General Procedure.—With an accurate procedure available for computing deflections of a beam when the moments are known, it is possible to set up a relatively simple procedure for handling deflections of bars subjected to axial loads as well as lateral loads, by successive approximations. In so far as the final deflections are concerned, the effect of lateral loads on the bar is the same as the effect of initial deflection of the bar from a straight line.

The following method of analysis is recommended for the general case:

(1) Divide the bar into a number of segments. Compute the deflections of the bar due to the lateral loads only, and add these deflections to the initial

deviations from a straight line. Let the total deflection with no axial loads be denoted by the symbol w_i.

(2) Guess at an assumed additional deflection, w_a, which is to represent the effect of the axial forces on the bar. Let the sum of w_a and w_i be denoted by w_0; that is,

$$w_0 = w_i + w_a \qquad (2)$$

(3) Compute the moments due to the axial loads on the bar, corresponding to the deflections w_0.

(4) Determine the deflections of the bar for the moments computed in step (3). Let these deflections be denoted by w'_a.

(5) Compare w'_a and w_a. If they are equal, w_a is the correct additional deflection of the bar, and w_0 is the correct total deflection of the bar. If they are not equal, repeat steps (2) to (5) until a desired measure of agreement is reached. One may take the values of w'_a in step (4) as a new set of assumed values of w_a, or one may modify these values in order to hasten the process and obtain agreement more rapidly between the assumed deflections and the resulting deflections.

It is necessary to point out that the procedure will work to advantage only when w'_a is a better approximation to the true additional deflections than w_a; in other words, the procedure works best when the sequence of successive approximations converges. It may not work at all when the sequence diverges or oscillates. One can formulate conditions that will insure convergence; but for practical purposes it will be evident that one either approaches a definite result or does not; and if the calculations approach a definite answer, it is the correct answer. Various "tricks" are possible in solving problems in which convergence is slow, or in which there is actually divergence of the results. However, such problems are not common. The writer does not wish to confuse this presentation with too elaborate a set of procedures for exceptional cases. It is sufficient to point out that if by any means w'_a and w_a can be made equal at all division points, one has the correct deflections. By trial, or by a systematic procedure, or by use of simultaneous equations, the two sets of values can always be made equal (even when the routine procedure of using the results in (4) as a new step (2) diverges), since one may take any arbitrary set of values of w_a.

Examples of the general procedure are given subsequently herein. Usually it is possible to obtain a good set of values of w_a if one has available a solution of the problem of pure buckling of the particular bar considered. For this reason a discussion of pure buckling will be given first.

Treatment of the Problem of Pure Buckling Without Lateral Loads.—Consider a bar subjected only to axial loads, without lateral load or initial deflection. Then the quantities w_i in step (1) of the general procedure are zero. The axial loads are to be determined so that an assumed set of deflections w_a corresponds to the same set of deflections w'_a, which means that the deflected bar is in a position of neutral equilibrium, and is on the point of reaching a position of stable (or possibly unstable) equilibrium which is different from the original undeflected position.

The special procedure becomes:

(*a*) Guess at a set of deflections w_a.

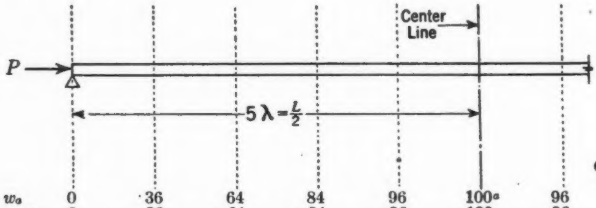
(*b*) Compute moments corresponding to the deflections w_a and an assumed set of values of the axial loads. It is convenient to consider generalized loads. That is, one considers the symbol P to represent all of the axial loads, or the system of such loads, acting on the bar. By assigning values to P, one assigns values to each of the individual loads which P represents.

(*c*) From the moments in step (*b*), compute deflections w'_a.

(*d*) Compare the deflections w'_a and w_a. If they are proportional—that is, if they can be made identically equal for a particular value of P, or for a particular set of values of the axial loads—there is a critical buckling load and the configuration of the bar corresponds to that load.

Again, certain questions may be raised regarding the convergence of a sequence of computations,[14] but these are beyond the scope of this paper; moreover, for most practical problems the difficulties do not arise.

Consider, for example, the calculation of the critical buckling load for a simply supported bar of constant cross section loaded at the ends with axial compressive forces. The calculations for an assumed parabolic deflection curve, symmetrical about the center line, are shown in Fig. 12. Only half the



														Common Factors
Assumed Deflections, w_a	0		36		64		84		96		100[a]		96	
Distributed Angle Changes	0		−36		−64		−84		−96		−100		−96	P/EI
Average Slope, Part (1)		330		294		230		146		50		−50		$P\lambda/EI$
Deflections, Part (1)	0		330		624		854		1,000		1,050		1,000	$P\lambda^2/EI$
Deflections, Part (2)	0		−3.0		−5.3		−7.0		−8.0		−8.3		−8.0	$P\lambda^2/EI$
Resultant Deflections, w'_a	0		327.0		618.7		847.0		992.0		1,041.7		992.0	$P\lambda^2/EI$
Ratio, $\frac{w_a}{w'_a}$	(0.1200)[b]		0.1101		0.1034		0.0992		0.0968		0.0960		0.0968	$EI/P\lambda^2$

(*a*) Average ratio, neglecting ratio for ends $= 0.1017 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 10.17 \frac{EI}{L^2}$

(*b*) Ratio of sums of deflections, $\frac{\Sigma w_a}{\Sigma w'_a} = 0.0998 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.98 \frac{EI}{L^2}$

(*c*) Best ratio by least squares, $\frac{\Sigma w_a w'_a}{\Sigma w'_a w'_a} = 0.0987 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.87 \frac{EI}{L^2}$

[a] Beam and deflections symmetrical about center line. [b] Ratios of end slopes.

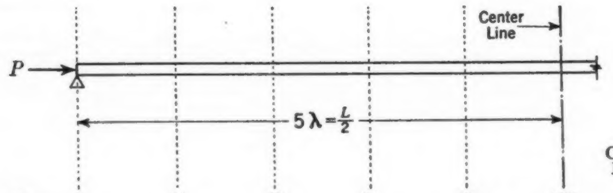
FIG. 12.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION, STARTING WITH ASSUMED PARABOLIC DEFLECTION CURVE

bar is considered since the structure and the deflections are symmetrical. It is seen that the ratio of w_a to w'_a is not constant; the different values of this ratio are recorded, and give the value of P required to produce equality of deflections at the particular points. A repetition of the calculation with new

[14] See, for example, "Zur Konvergenz des Engesser-Vianello-Verfahrens," by A. Schleusner, Berlin, 1938.

values of w_a, equal to, or proportional to, the values of w'_a shown in Fig. 12, would give more nearly uniform ratios. The best value of the critical load may be taken as the average of the ratios, or as some weighted average; in Fig. 12, three different values of the critical load, computed in different ways, are reported. From similar calculations, it is the writer's conclusion that in most cases a reasonably good approximation to the critical load is the ratio of the sums of the ordinates to the curves of w_a and w'_a.

A more uniform set of ratios with a correspondingly better approximation to the critical load is given with a curve that more nearly approaches the true buckling configuration. In Fig. 13 a set of values is assumed for w_a approximately proportional to the values of w'_a determined in Fig. 12. The result is practically exact. In both Figs. 12 and 13, the true value of the critical load is $\pi^2 E I/L^2$; or, 9.870 $E I/L^2$.



												Common Factors
Assumed Deflections, w_a	0		31		59		81		95		100	
Distributed Angle Changes	0		−31		−59		−81		−95		−100	$P/E I$
Average Slope, Part (1)		316		285		226		145		50		$P \lambda/E I$
Deflection, Part (1)	0		316		601		827		972		1,022	$P \lambda^2/E I$
Deflection, Part (2)	0		−2.6		−4.9		−6.8		−7.9		−8.3	$P \lambda^2/E I$
Resultant Deflection, w'_a	0		313.4		596.1		820.2		964.1		1,013.7	$P \lambda^2/E I$
Ratio, $\frac{w_a}{w'_a}$	..		0.0989		0.0990		0.0987		0.0985		0.0986	$E I/P \lambda^2$

$$\text{Average ratio} = 0.0988 \frac{E I}{P \lambda^2}$$

$$\text{Ratio of sums} = 0.0987 \frac{E I}{P \lambda^2}$$

$$\therefore \quad P_{cr} = 0.0987 \frac{E I}{\lambda^2} = 9.87 \frac{E I}{L^2}$$

FIG. 13.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION

Obviously it is possible to find different patterns of deflections corresponding to different values of critical loads for the same bar. In general, only the lowest critical load is of significance as far as pure buckling is concerned, since the higher loads must correspond to essentially unstable positions of equilibrium; but if an initial deflection curve is assumed that contains no components of the configuration corresponding to the lowest critical buckling load, the lowest load cannot be obtained from this procedure (nor would it be obtained from any other available procedure, such as methods involving minimum of energy). Such a situation would follow from the assumption of a deflection curve anti-symmetrical about the center line for the beam in Fig. 12. One would reject such a curve intuitively for this problem. Yet in an unusual case, it might be possible that a designer may reject, unthinkingly, the configuration that corresponds to the lowest buckling load. An example of such a case is shown subsequently in Fig. 20.

Ordinarily, convergence of several different sequences of computations involving different shapes of assumed deflection curves, to the same final shape, would be a sufficient indication that the designer had reached the configuration corresponding to the lowest critical load. In some cases, however, the convergence of a sequence of computations may be very slow; this will be so when the next higher critical load differs only slightly from the lowest critical load. Methods of handling such problems can be derived but are beyond the scope of the present paper.

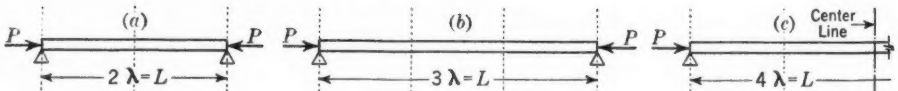
Determination of Maximum and Minimum Values for the Critical Load.—In general, the lowest critical buckling load must have a value between the limits defined by the smallest and largest values of the ratio of w_a to w'_a, when all values of w_a and w'_a are positive. One can reason as follows to justify this rule: If every point on the derived deflection curve lies outside of every point on the assumed deflection curve, the load must be greater than the load required to produce neutral equilibrium, since the bar is tending to deflect even farther away from its original straight configuration than assumed. This means that the initial straight configuration is now an unstable position of equilibrium. On the other hand, if every point on the derived deflection curve lies between the original straight configuration and the assumed configuration, then the load must be less than the load required to produce neutral equilibrium. Evidently, in this case, the undeflected position is a position of stable equilibrium; but the two conditions described correspond to the maximum and the minimum values of the ratio of w_a to w'_a. Therefore the critical buckling load must be between these limits. The rule is important for practical purposes; the designer can readily detect between what limits the buckling load must lie.

In using this rule to bound the value of the critical load, it must be remembered that the structure set up for analysis is not exactly the same as the structure it represents, although with a reasonably large number of divisions the two are closely similar. The process of dividing the bar into segments is equivalent to substituting for it a slightly different structure. This becomes evident if the buckling load is computed for a bar divided into only two segments, as in Fig. 14(*a*).

In certain cases the foregoing rule is inapplicable. Care must be taken in using it when axial loads are applied other than at the ends of a bar. Also, the rule would be misleading in such cases where the lowest critical load corresponds to a deflection curve that has both positive and negative deflections, whereas the next higher critical load might correspond to a deflection curve with only positive ordinates.

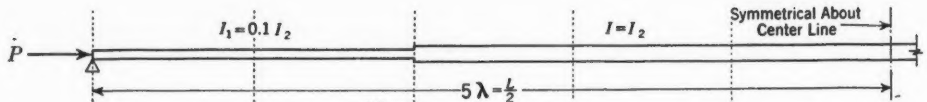
Illustrative Problems for Pure Buckling.—The problems shown in Figs. 14, 15, and 16 illustrate further uses of the procedure for computing the critical load for a beam subjected to pure buckling.

The effect of taking different numbers of segments in the length of the bar is illustrated in Fig. 14, for a simply supported bar of uniform section subjected to end thrust. The error in the buckling load computed by the approximate procedure described herein, compared with the exact buckling load, is 2.74% for two segments, 0.52% for three segments, and 0.16% for four segments, in the full length of the bar.



	(a)			(b)				(c)			Common Factors
Assumed Deflections, w_a	0	2	0	0	1	1	0	0	707	1,000	
Distributed Angle Changes	0	−2	0	0	−1	−1	0	0	−707	−1,000	$\frac{P}{EI}$
Average Slope, Part (1)	1	−1		1	0	−1		1,207	500		$\frac{P\lambda}{EI}$
Deflection, Part (1)	0	1	0	0	1	1	0	0	1,207	1,707	$\frac{P\lambda^2}{EI}$
Deflection, Part (2)	0	$-\frac{2}{12}$	0	0	$-\frac{1}{12}$	$-\frac{1}{12}$	0	0	−59	−83	$\frac{P\lambda^2}{EI}$
Resultant Deflection, w'_a	0	$\frac{10}{12}$	0	0	$\frac{11}{12}$	$\frac{11}{12}$	0	0	1,148	1,624	$\frac{P\lambda^2}{EI}$
Ratio, $\frac{w_a}{w'_a}$		2.40			1.091	1.091			0.6159	0.6158	$\frac{EI}{P\lambda^2}$
P_{cr}	$2.40\frac{EI}{\lambda^2} = 9.60\frac{EI}{L^2}$			$1.091\frac{EI}{\lambda^2} = 9.819\frac{EI}{L^2}$				$0.6159\frac{EI}{\lambda^2} = 9.854\frac{EI}{L^2}$			
Percentage Error in Terms of Exact Value, $P_{cr} = 9.870\frac{EI}{L^2}$,	2.74			0.52				0.16			

FIG. 14.—EFFECT ON BUCKLING LOAD OF TAKING DIFFERENT NUMBERS OF SEGMENTS FOR BAR OF CONSTANT CROSS SECTION



(a)

												Common Factors
Assumed Deflection, w_a	0		51.35		80.36		91.10		97.75		100.00	
Distributed Angle Changes	0		−513.5		−803.6 ¦ −80.36		−91.10		−97.75		−100.00	$P/E\,I_2$
Equivalent Concentrated Angle Changes			−494.88		−404.90		−90.76		−97.38		−99.63	$P\,\lambda/E\,I_2$
Average Slope		1,137.74		642.86		237.96		147.20		49.82		$P\,\lambda/E\,I_2$
Resultant Deflection, w'_a	0		1,137.74		1,780.60		2,018.56		2,165.76		2,215.58	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$			0.04513		0.04513		0.04513		0.04513		0.04513	$E\,I_2/P\,\lambda^2$

$$P_{cr} = 0.04513\,\frac{E\,I_2}{\lambda^2} = 4.513\,\frac{E\,I_2}{L^2}$$

(b)

												Common Factors
Assumed Deflection, w_a	0		36		64		84		96		100	
Distributed Angle Changes	0		−360		−640 ¦ −64		−84		−96		−100	$P/E\,I_2$
Equivalent Concentrated Angle Changes			−353.3		−312.3		−83.3		−95.3		−99.3	$P\,\lambda/E\,I_2$
Average Slopes		893.9		540.6		228.3		145.0		49.7		$P\,\lambda/E\,I_2$
Resultant Deflection, w'_a	0		893.9		1,434.5		1,662.8		1,807.8		1,857.5	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$			0.0403		0.0446		0.0505		0.0531		0.0538	$E\,I_2/P\,\lambda^2$

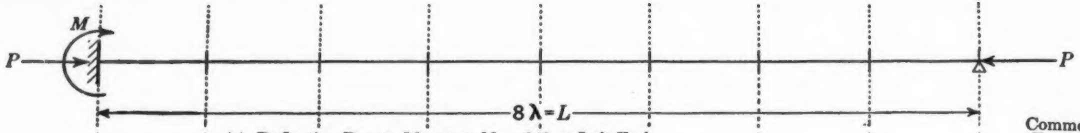
$$\sim P_{cr} = \frac{\Sigma\,w_a}{\Sigma\,w'_a} = 4.91\,\frac{E\,I_2}{L^2}$$

(c)

							Common Factors
Assumed Deflection, w_a	0	50	80	91	97	100	
Resultant Deflection, w'_a	0	1,120.8	1,758.3	1,995.5	2,142.1	2,191.9	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$		0.0446	0.0455	0.0456	0.0453	0.0456	$E\,I_2/P\,\lambda^2$

$$\sim P_{cr} = \frac{\Sigma\,w_a}{\Sigma\,w'_a} = 4.54\,\frac{E\,I_2}{L^2}$$

FIG. 15.—BUCKLING OF A BAR WITH CHANGE IN SECTION



(a) Deflection Due to Moment M = 240 at Left End

																		Common Factors
Moments	240		210		180		150		120		90		60		30		0	
Distributed Angle Changes	−240		−210		−180		−150		−120		−90		−60		−30		0	$1/E\,I$
Assumed Average Slope, Part (1)		500		290		110		−40		−160		−250		−310		−340		$\lambda/E\,I$
Trial Deflection, Parts (1) and (2)	0		500		790		900		860		700		450		140		−200	$\lambda^2/E\,I$
Linear Correction to Deflection	0		25		50		75		100		125		150		175		200	$\lambda^2/E\,I$
Resultant Deflection, w_a	0		525		840		975		960		825		600		315		0	$\lambda^2/E\,I$

$$\text{Slope at left end} = (525 + 115)\frac{\lambda}{E\,I} = 640\frac{\lambda}{E\,I}$$

(b) Deflections Due to End Loads with Assumed Deflections

																		Common Factors
Assumed Deflection, w_a	0		32		115		212		286		307		259		148		0	a
Distributed Angle Changes	0		−32		−115		−212		−286		−307		−259		−148		0	$P\,a/E\,I$
Assumed Average Slope, Part (1)		585		553		438		226		−60		−367		−626		−774		$P\,\lambda\,a/E\,I$
Trial Deflection, Part (1)	0		585		1,138		1,576		1,802		1,742		1,375		749		−25	$P\,\lambda^2\,a/E\,I$
Deflection, Part (2)	0		−3		−10		−18		−24		−26		−22		−12		0	$P\,\lambda^2\,a/E\,I$
Linear Correction to Deflection	0		3		6		9		13		16		19		22		25	$P\,\lambda^2\,a/E\,I$
Intermediate Deflection, w_b	0		585		1,134		1,567		1,791		1,732		1,372		759		0	$P\,\lambda^2\,a/E\,I$

$$\text{End slope} = (585 + 3)\frac{P\,\lambda\,a}{E\,I} = 588\frac{P\,\lambda\,a}{E\,I}$$

(c) Effect of Making Slope at Left End Zero, Moment at Left End $= -\dfrac{588\dfrac{P\,\lambda\,a}{E\,I}}{640\dfrac{\lambda}{E\,I}}(240) = -220.5\,P\,a$

										Common Factors
$-\dfrac{220.5\,P\,a}{240}\,w_a$	0	−482	−772	−896	−882	−758	−551	−289	0	$P\,\lambda^2\,a/E\,I$
Resultant Deflection, w'_a	0	103	362	671	909	974	821	470	0	$P\,\lambda^2\,a/E\,I$
Ratio, $\dfrac{w_a}{w'_a}$		0.310	0.318	0.316	0.314	0.315	0.315	0.315		$E\,I/P\,\lambda^2$

$$P_{cr} = 0.315\frac{E\,I}{\lambda^2} = 20.16\frac{E\,I}{L^2}$$

FIG. 16.—BUCKLING OF BAR FIXED AT ONE END

In Fig. 15(*a*), a solution is given for the buckling of a bar consisting of parts of constant but different moment of inertia. Because of the abrupt change in moment of inertia there is a discontinuity in the values of the angle changes in the bar. The result obtained with only five divisions in the half-length of the bar is

$$P_{cr} = 4.51 \frac{E\, I_2}{L^2} \quad \text{.......................(3)}$$

which compares with $4.50 \frac{E\, I_2}{L^2}$ given by Professor Timoshenko[15] as the "exact" value of the critical load. It should be remembered that several trials were necessary before as uniform a set of ratios as is shown in Fig. 15(*a*) was obtained; but the intermediate work can be done without refinements and the final result obtained fairly rapidly. For practical purposes it would not be necessary to go so far. For example, Figs. 15(*b*) and 15(*c*) might contain all the calculations required in most cases, where even the first step, starting with an assumed parabolic deflection curve, would be adequate for almost any practical problem.

In a similar manner, other problems involving variations in moment of inertia along the length of the bar may be solved. Where the variation is smooth (that is, without abrupt changes) the relatively simple modified procedure which does not require calculation of "equivalent" concentrated angle changes may be used.

The solution of the problem of buckling of a bar fixed at one end and simply supported at the other is shown in Fig. 16. The problem is solved by adding to a simply supported bar an end moment to annul the rotation at one end of the bar. The problem might also have been solved by dealing with a cantilever beam acted on by a direct thrust, and adding the effect of a lateral load at the end in order to make the deflection at the end zero. The results would have been exactly the same.

The procedure used in Fig. 16 may be outlined as follows:

(*a*) Find the deflections and end rotation of a simply supported bar due to a moment applied at one end. Denote the deflections by w_s.

(*b*) Assume a deflection curve for the bar fixed at one end and simply supported at the other. Denote the deflections by w_a. Compute the moments in the bar due to the direct loads and the deflections w_a. One may also include assumed moments to account in some measure for the effect of fixing the one end of the bar. In general, it would be desirable to include such "indeterminate" moments, although in Fig. 16 they were omitted.

(*c*) Compute the deflections w_b and the end rotation corresponding to the moments in step (*b*). If the end rotation is not zero, add such a moment as would be required to make it zero. This involves adding deflections also, proportional to w_s. Denote the resultant deflections by the symbol w'_a.

(*d*) Compare w_a and w'_a, as in the procedure described previously for determining buckling loads for statically determinate bars. If w_a and w'_a are similar, one has the correct shape of the deflection curve, and one can obtain the critical load. If w_a and w'_a are not similar, one may repeat steps (*b*) to (*d*)

[15] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 128–131.

as many times as necessary, until one obtains a sufficiently good value of the critical load.

A procedure similar to the foregoing may be developed for other statically indeterminate beams or columns.

Note that in Fig. 16(*a*) the moment diagram and the angle change diagram are linear; therefore it was not necessary to compute part (2) of the deflections, as explained in section II of this paper. In Fig. 16(*b*) a common factor a is indicated for the deflections in order to make it clear that the end moment in Fig. 16(*c*) depends on the deflections. The final value of the critical load is practically exact.[16]

Illustrative Problems, Combined Axial and Lateral Loads.—When lateral loads act on a beam together with an end thrust, the effect of the end thrust is to produce additional deflections and additional moments beyond those produced by the lateral loads alone. The additional deflections are governed by the deflection due to the lateral load alone, and by the ratio of the axial loads to the critical value of the axial loads.

For the first step in the general procedure of solving such problems it is necessary to assume a set of values of the additional deflection, w_a. As a convenient approximation for the first trial value of w_a it is desirable to take w_a as follows:

$$w_a = \frac{1}{\frac{P_{cr}}{P} - 1} w_i \qquad (4)$$

in which P_{cr} is the magnitude of the critical buckling load, P is the magnitude of the actual load, and w_i is the sum of the initial deflection and the deflection due to the lateral load alone. When w_i is of the same shape as the deflection curve corresponding to the lowest critical buckling load, the value of w_a given by Eq. 4 will be exact.[17] In other cases, it hastens the convergence toward the correct value of w_a if w_a is assumed as suggested.

The calculations for a simply supported bar subjected to end thrusts and uniform lateral load are shown in Fig. 17. The values of w_i for the uniform load are computed first. The value of P_{cr} for the bar can be taken from Fig. 12. Then with the given load, $P = 0.02 \frac{E\,I}{\lambda^2}$, and the critical load, $P_{cr} = 0.0987 \frac{E\,I}{\lambda^2}$, one finds from Eq. 4 the following result:

$$w_a = 0.254\, w_i \qquad (5)$$

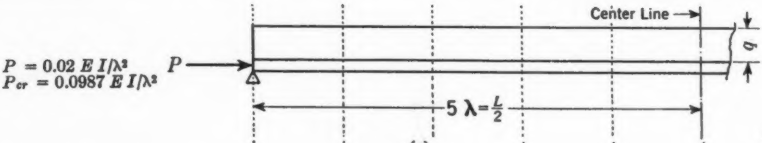
With this value of w_a, the computations in Fig. 17(*b*) lead to a set of values of w'_a which are practically equal to those assumed. If further refinement is desired one can repeat the calculations. One may also deal with additions to the values of w_a already assumed and obtain additions or corrections to w'_a; but in this problem no further computations appear to be necessary, and one may conclude that under the given conditions the effect of the axial load is to cause an apparent increase in the maximum moment due to the lateral loads alone of about 26%.

[16] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 88–89.

[17] See, for example, "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), pp. 576–676, especially pp. 618–619. Note difference in notation, however.

Similar calculations are shown in Fig. 18 for a bar subjected to a moment at one end combined with direct thrust. Here the first approximation is not nearly so close to the final answer since the deflection curve due to the moment alone differs considerably from the configuration corresponding to the critical buckling load.

The procedure described here is applicable also to problems in which P is negative; that is, where axial tensions instead of compressions act on the bar. In such cases, however, the effect of the end tension is generally to reduce the deflections due to the lateral loads or initial eccentricities only. The same



(a)

												Common Factors
Moments Due to Lateral Load	0		36		64		84		96		100	$q\lambda^2/8$
Distributed Angle Changes	0		−36		−64		−84		−96		−100	$q\lambda^2/8EI$
Average Slope, Part (1)		330		294		230		146		50		$q\lambda^3/8EI$
Deflection, Part (1)	0		330		624		854		1,000		1,050	$q\lambda^4/8EI$
Deflection, Part (2)	0		−3		−5		−7		−8		−8	$q\lambda^4/8EI$
Deflection, w_i	0		327		619		847		992		1,042	$q\lambda^4/8EI$

(b)

Assumed Deflection, w_a from Eq. 4	0		83		157		216		252		265	$q\lambda^4/8EI$
$w_0 = w_a + w_i$	0		410		776		1,063		1,244		1,307	$q\lambda^4/8EI$
Moments Due to P	0		8.20		15.52		21.26		24.88		26.14	$q\lambda^2/8$
Distributed Angle Changes	0		−8.20		−15.52		−21.26		−24.88		−26.14	$q\lambda^2/8EI$
Average Slope, Part (1)		82.93		74.73		59.21		37.95		13.07		$q\lambda^3/8EI$
Deflection, Part (1)	0		82.93		157.66		216.87		254.82		267.89	$q\lambda^4/8EI$
Deflection, Part (2)	0		−0.68		−1.29		−1.77		−2.07		−2.18	$q\lambda^4/8EI$
Resultant Deflection, w'_a	0		82		156		215		253		266	$q\lambda^4/8EI$

FIG. 17.—DEFLECTION OF A BAR SUBJECTED TO UNIFORM LOAD AND END THRUST

general procedure may be used. The value of w_a suggested in Eq. 4 will be negative, since, if the axial tensions are denoted by T, one has the result

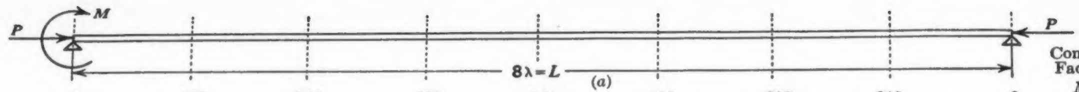
$$w_a = -\frac{1}{\frac{P_{cr}}{T} + 1} w_i \qquad (6)$$

A difficulty arises in problems where T is numerically greater than the value of the lowest critical buckling load. In such cases the sequence of computations will oscillate, and will not converge. In general, each assumed value of w_a will lead to a value of w'_a which will be farther from the true configuration than w_a if w_a is not correctly chosen equal to its true value. Methods of solving such problems can be developed, however, and in general one can arrive eventually at reasonably good results since the effect of the end tensions can never be to produce greater deflections than w_i except at a few points. Further discussion of problems such as these will not be given in the present paper.

Buckling Due to Axial Loads Applied at Intermediate Points Along the Length of a Bar.—The problems previously treated herein concern axial loads applied at the ends of a bar. Bars with axial loads applied at interior points are considered in Figs. 19 and 20. In Fig. 19, the left part of the bar is in compression

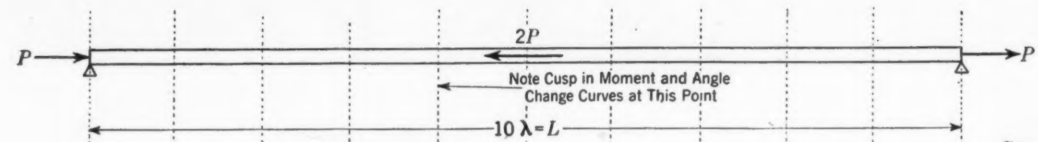
$P = 0.10\, E\,I/\lambda^2$

$P_{cr} = 9.87\, E\,I/L^2 = 0.1542\, E\,I/\lambda^2$



										Common Factors
w_i from Fig. 16(a)	0	525	840	975	960	825	600	315	0	$\frac{M\lambda^2}{240\,E\,I}$
(b)										
Assumed Deflection, w_a from Eq. 4	0	969	1,550	1,799	1,771	1,522	1,107	581	0	$\frac{M\lambda^2}{240\,E\,I}$
$w_0 = w_a + w_i$	0	1,494	2,390	2,774	2,731	2,347	1,707	896	0	$\frac{M\lambda^2}{240\,E\,I}$
Moments due to P	0	149.4	239.0	277.4	273.1	234.7	170.7	89.6	0	$\frac{M}{240}$
Distributed Angle Changes	0	−149.4	−239.0	−277.4	−273.1	−234.7	−170.7	−89.6	0	$\frac{M}{240\,E\,I}$
Assumed Average Slope, Part (1)		750.0	600.6	361.6	84.2	−188.9	−423.6	−594.3	−683.9	$\frac{M\lambda}{240\,E\,I}$
Trial Deflection, Part (1)	0	750.0	1,350.6	1,712.2	1,796.4	1,607.5	1,183.9	589.6	−94.3	$\frac{M\lambda^2}{240\,E\,I}$
Deflection, Part (2)	0	−12.5	−19.9	−23.1	−22.8	−19.6	−14.2	−7.5	0	$\frac{M\lambda^2}{240\,E\,I}$
Linear Correction to Deflection	0	11.8	23.6	35.4	47.2	58.9	70.7	82.5	94.3	$\frac{M\lambda^2}{240\,E\,I}$
Resultant Deflection, w'_a	0	749.3	1,354.3	1,724.5	1,820.8	1,646.8	1,240.4	664.6	0	$\frac{M\lambda^2}{240\,E\,I}$
(c) Final Deflections w_a which Lead to Same Deflections w'_a										
	0	720	1,319	1,705	1,826	1,672	1,271	685	0	$\frac{M\lambda^2}{240\,E\,I}$
(d) Moments in Beam										
Due to M Alone	240	210	180	150	120	90	60	30	0	$\frac{M}{240}$
Due to Thrust	0	124.5	215.9	268.0	278.6	249.7	187.1	100.0	0	$\frac{M}{240}$
Total Moment	240.0	344.5	395.9	418.0	398.6	339.7	247.1	130.0	0	$\frac{M}{240}$
Total Moment	1.000	1.394	1.650	1.742	1.661	1.415	1.030	0.542	0	M

FIG. 18.—DEFLECTION OF A BAR SUBJECTED TO END MOMENT AND DIRECT COMPRESSION



(a) Assumed Parabolic Deflection Curve

												Common Factors
Assumed Deflection, w_a	0	375	667	875	1,000	1,042	1,000	875	667	375	0	a
Moment	0	175	267	275	200	−42	−200	−275	−267	−175	0	$P\,a$
Angle Changes	0	−175	−267	−275	−200	+42	+200	+275	+267	+175	0	$P\,a/E\,I$
Equivalent Concentrated Angle Changes	0	−168	−260	−268	−172	35	193	268	260	168	0	$P\,a\,\lambda/E\,I$
Assumed Average Slope		404	236	−24	−292	−464	−429	−236	32	292	460	$P\,a\,\lambda/E\,I$
Trial Deflection	0	404	640	616	324	−140	−569	−805	−773	−481	−21	$P\,a\,\lambda^2/E\,I$
Linear Correction to Deflection	0	2	4	6	8	11	13	15	17	19	+21	$P\,a\,\lambda^2/E\,I$
Resultant Deflection, w'_a	0	406	644	622	332	−129	−556	−790	−756	−462	0	$P\,a\,\lambda^2/E\,I$

(b) Final Deflection Curve Leading to a Negative Value of P_{cr}

w_a	0	226	460	710	1,000	1,323	1,547	1,544	1,255	698	0	a
Moment	0	26	60	110	200	−323	−747	−944	−855	−498	0	$P\,a$
w'_a	0	732	1,491	2,312	3,245	4,271	4,981	4,964	4,027	2,256	0	$-P\,a\,\lambda^2/E\,I$
Ratio $\frac{w_a}{w'_a}$		0.309	0.308	0.307	0.308	0.310	0.311	0.311	0.312	0.309		$-E\,I/P\,\lambda^2$

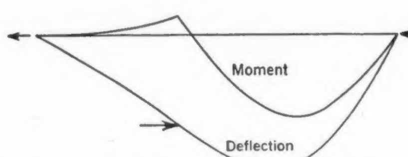
$$P_{cr} = -\,0.310\,\frac{E\,I}{\lambda^2} = -\,31.0\,\frac{E\,I}{L^2}$$

(c) Final Deflection Curve Leading to a Positive Value of P_{cr}

w_a	0	480	839	1,009	1,000	899	749	570	388	197	0	a
Moment	0	280	439	409	200	101	51	30	12	3	0	$P\,a$
w'_a	0	1,078	1,885	2,270	2,260	2,027	1,688	1,296	874	439	0	$P\,a\,\lambda^2/E\,I$
Ratio $\frac{w_a}{w'_a}$		0.445	0.445	0.444	0.442	0.444	0.444	0.440	0.444	0.449		$E\,I/P\,\lambda^2$

$$P_{cr} = 0.444\,\frac{E\,I}{\lambda^2} = 44.4\,\frac{E\,I}{L^2}$$

(d) Deflections and Moments for Fig. 19(b)



(e) Deflections and Moments for Fig. 19(c)

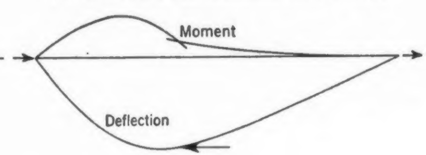
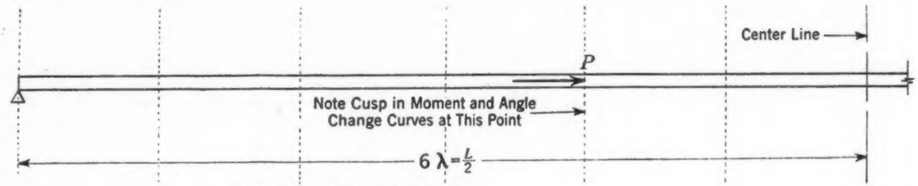


Fig. 19.—Buckling of a Bar Subjected to Axial Tension and Compression



(a) Symmetrical Configuration

														Common Factors
Assumed Deflection, w_a	0		...		...		...		c		$c+1{,}414$		$c+2{,}000$	a
Moment	0		0		0		0		0		1,414		2,000	$P\,a$
Angle Changes	0		0		0		0		0		−1,414		−2,000	$P\,a/E\,I$
Equivalent Concentrated Angle Changes	0		0		0		0		−270		−1,345		−1,902	$P\,a\,\lambda/E\,I$
Average Slope		2,566		2,566		2,566		2,566		2,296		951		$P\,a\,\lambda/E\,I$
Resultant Deflection, w'_a									c'		$c'+2{,}296$		$c'+3{,}247$	$P\,a\,\lambda^2/E\,I$
Ratio, $\frac{w_a}{w'_a}$, if $\frac{c}{c'}$ has same value as ratio:											0.6159		0.6160	$E\,I/P\,\lambda^2$

$$P_{cr} = 0.6159\,\frac{E\,I}{\lambda^2}$$

Deflection, w'_a	0	2,566	5,132	7,698	10,264	12,560	13,511	$P\,a\,\lambda^2/E\,I$

(b) Antisymmetrical Configuration

Assumed Deflection, w_a	0		...		....		....		3,948		2,424		0	a
Moment	0		658		1,316		1,974		2,632		1,766		0	$P\,a$
Angle Changes	0		−658		−1,316		−1,974		−2,632		−1,766		0	$P\,a/E\,I$
Equivalent Concentrated Angle Changes			−658		−1,316		−1,974		−2,415.5		−1,691		0	$P\,a\,\lambda/E\,I$
Assumed Average Slope		3,500		2,842		1,526		−148		−2,863.5		−4,554.5		$P\,a\,\lambda/E\,I$
Trial Deflection	0		3,500		6,342		7,868		7,420		4,556.5		2.0	$P\,a\,\lambda^2/E\,I$
Linear Correction to Deflection	0		−0.3		−0.7		−1.0		−1.3		−1.7		−2.0	$P\,a\,\lambda^2/E\,I$
Resultant Deflection, w'_a	0		3,500		6,341		7,867		7,419		4,555		0	$P\,a\,\lambda^2/E\,I$
Ratio, $\frac{w_a}{w'_a}$									0.5321		0.5322			$E\,I/P\,\lambda^2$

$$P_{cr} \doteq 0.5322\,\frac{E\,I}{\lambda^2}$$

FIG. 20.—BUCKLING OF A BAR WITH COMPRESSION IN MIDDLE THIRD OF LENGTH

and the right part in tension. The point of application of the interior load is assumed to deflect with the bar; consequently if the load point deflects shears must be applied at the ends of the bar for equilibrium. Since there will be a cusp, or discontinuity in slope of the angle-change diagram at the point of application of the interior load, the procedure used is to write the equivalent concentrated angle changes instead of making the correction that can be made for a smooth angle change curve. In Fig. 19(a) a symmetrical parabolic deflection curve is assumed first. One finds a peculiar result: Some of the resulting deflections are negative. If these deflections are taken as a new deflection curve, and the process repeated, eventually one comes to the result shown in Fig. 19(b) where, apparently, the critical load is negative; but this merely indicates a situation in which the left part of the bar is in tension and the right part in compression. It is reasonable that the buckling load should be less for this arrangement of loads since a longer part of the bar is thereby subjected to compression. The final result for the original problem is shown in Fig. 19(c). It may be obtained by repeated trials, but not by a process in which each new configuration is the result obtained from a previous assumed configuration, unless the starting point is a configuration not containing any appreciable component of the type obtained in Fig. 19(b). The shapes of the final deflection curves and the moment diagrams corresponding thereto are shown in Fig. 19(d).

A bar subjected to two opposing loads applied at the third points is illustrated by Fig. 20. An exact solution for this problem is available.[18] The problem is given not only to illustrate the procedure for an unusual case, but also to show what can happen when care is not taken to insure that components of deflection corresponding to the lowest critical buckling load are present in the assumed deflection curve. The loads are assumed to be applied on the axis of the bar even when the bar deflects.

A symmetrical deflection of the bar is shown in Fig. 20(a). The deflections outside of the region subjected to compression are immaterial in a consideration of the critical buckling load. It will be noted that the critical load is the same as in Fig. 14(c); but some care is necessary in obtaining the proper value of c, the unknown constant part of all the deflections in the region considered. For the final deflection curve c can be obtained easily by taking the complete deflection curve for w'_a and repeating the calculations; but for intermediate steps, c can be chosen as having any value, which complicates the problem of placing a limit on the critical load. Obviously there should be no distortion in the region outside of the central part of the bar, however, and therefore one can always make a fair estimate of the situation in this case.

In Fig. 20(b) an antisymmetrical deflection is assumed, and the corresponding critical load is calculated. Here again, the deflections outside of the region subject to compression do not enter into the finding of the critical load. It is of interest and importance that the critical load corresponding to the antisymmetrical deflection is lower than that corresponding to the symmetrical configuration for the arrangement of loads chosen. The bar would actually tend to buckle by more or less of a rotation of the central section. However,

[18] "Über die Knickung eines Balkens durch Längskräfte," by O. Blumenthal, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 17, 1937, pp. 232–244, especially pp. 234–239.

this would not have been discovered if only symmetrical deflection curves had been assumed.

PART IV.—CONCLUDING REMARKS

Treatment of Large Deflections.—In all of the problems discussed herein the fundamental relation between deformation and moment has been implicitly assumed to be of the following type:

$$\frac{d^2y}{dx^2} = -\frac{M}{E\,I} \qquad (7)$$

in which y is the deflection, positive downward. Where deflections are large, Eq. 7 is only approximately correct. One should replace it by the relation:

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = -\frac{M}{E\,I} \qquad (8a)$$

or

$$\frac{d^2y}{dx^2} = -\frac{M}{E\,I}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad (8b)$$

The use of the exact relation offers no serious difficulties. In computing deflections from it one must assume the values of the deflections first, and determine the "angle changes" from modified values of $-\frac{M}{E\,I}$ (by multiplying by a function of the slopes at various points along the bar). One computes the deflections by a series of successive approximations, in which each step is similar to the various procedures outlined in the paper. However, it is not often necessary to consider such refinements.

Further Applications.—The procedure described herein is applicable to many other problems, since it permits a relatively simple and accurate numerical integration of a class of differential equations.

For example, the problem of a beam on elastic supports can be solved by first assuming a set of deflections, then determining the forces acting on the beam, with the consequent moments and angle changes. From the angle changes, the deflections can be computed. If these are the same as the assumed deflections, the problem is solved. If they are different, the process must be repeated.

The general procedure may also be modified so as to solve the problem of determining the natural period of vibration of a beam, or the critical speed of a shaft. Problems of this kind have been solved previously by similar procedures.[5,6] The use of the present modification is to produce a more accurate solution with generally less effort.

Conclusion.—The numerical procedure described herein permits a simple and rapid calculation of deflections of beams and columns and of critical buckling loads for columns with a high degree of accuracy. The method can be extended to other problems of the same mathematical nature.

ACKNOWLEDGMENT

The method of analysis described herein was developed as part of the writer's work in the Engineering Experiment Station of the University of

Illinois, Urbana, Ill. Particular acknowledgment is due Harold Crate, Research Graduate Assistant in Civil Engineering, for assistance in making and checking the calculations, and for extended studies of the procedure.

APPENDIX

Derivation of Formulas for Equivalent Concentrations

In Fig. 5(a) the origin of coordinates is at a, with x positive to the right, and q represents the magnitude of the load at any point, positive upward. Let $z = \frac{x}{\lambda}$, in order to obtain a dimensionless coordinate, and consider the curve of loading with ordinates a, b, c at $z = 0, 1, 2$, respectively, to be a second-degree function of z or of x.

It can be readily verified that Eq. 9 represents a general second-degree function of z having the required values $q = a, b, c$ at $z = 0, 1, 2$, respectively:

$$q = \tfrac{1}{2}\, a\, (z - 1)\, (z - 2) - b\, z\, (z - 2) + \tfrac{1}{2}\, c\, z\, (z - 1) \qquad (9)$$

Then, from statics the equivalent concentrated loads R_{ab} and R_{ba} are determined by the equations:

$$R_{ab} + R_{ba} = \lambda \int_0^1 q\, dz \qquad (10a)$$

and

$$R_{ba} = \lambda \int_0^1 z\, q\, dz \qquad (10b)$$

Evaluation of the integrals yields the results

$$R_{ba} = \frac{\lambda}{24}\,(3\,a + 10\,b - c) \qquad (11a)$$

and

$$R_{ab} = \frac{\lambda}{24}\,(7\,a + 6\,b - c) \qquad (11b)$$

By analogy, one finds the value of R_{bc}:

$$R_{bc} = \frac{\lambda}{24}\,(3\,c + 10\,b - a) \qquad (12)$$

from which is readily obtained the value of R_b:

$$R_b = R_{ba} + R_{bc} = \frac{\lambda}{12}\,(a + 10\,b + c) \qquad (13)$$

Similar formulas can be written for a loading curve of higher degree in x or z in terms of ordinates at more points. One may also derive expressions for R in terms of differences or of central differences by expressing q in such terms. Furthermore, one may develop corresponding equations when the segments into which the loading curve is divided are not of equal length. However, for practical purposes Eqs. 11 and 13 are all that are generally needed.

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PAPERS

MECHANICS OF CREEP FOR STRUCTURAL ANALYSIS

By Joseph Marin,[1] Assoc. M. Am. Soc. C. E.

Synopsis

Under certain conditions in a stressed structural material, a continuous plastic deformation occurs with time for a constant stress value. This deformation is called "creep." The importance of creep in a particular design depends upon various factors as, for example, the material considered, the allowable stress used, the temperature of operation, and the desirability of maintaining a small deformation. For this reason the designer must consider the effect of creep in many applications, such as occurs in the turbine, oil-refining and automotive industries. In cases where high temperatures occur, changes in the stress distribution and deformations are sometimes so great that the need for considering effects of creep has been realized for several years. For normal temperatures, on the other hand, little has been done to determine the influence of creep as a factor in structural design. This paper is an attempt to analyze this problem. In developing methods of analysis for statically indeterminate structures it was necessary first to select a creep-stress law that could be used in place of Hooke's law. For this purpose a new creep-stress law is proposed in this paper. With a basic creep-stress relation for simple tension, theories for bending stresses and deflections in beams are developed. These theories form the basis for the slope-deflection and moment-distribution procedures developed for the analysis of statically indeterminate structures in the case of creep.

The importance of creep in structural analysis is twofold: First, there is the possibility that the creep deflection might be excessive; and second, there is, in the case of statically indeterminate structures, a change in the values of reactions and resisting moments when Hooke's law is replaced by a creep law. The significance of the creep deflection or the change in stress distribution in

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by October 1, 1942.

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a particular design naturally will depend upon the material considered. Calculations indicate that in some cases a consideration of creep is important.

Some preliminary tests are reported in this paper in support of theories developed. It should be emphasized, however, that these theories will give only approximations, since the creep law is an approximate one. It must also be noted that there is great need for experimental work in this new field of study.

The importance of creep in structural analysis is particularly significant if stresses beyond the yield point are permitted or if such stresses are produced due to overloading. This is the case because there is evidence of creep for such stresses. A consideration of creep is necessary, therefore, for some structural materials and applications at both normal and elevated temperatures. Its importance is also magnified in view of the recent interest shown in designing structural parts stressed to the plastic range.

Introduction

When they are subjected to stress, engineering materials sometimes develop plastic deformations with time, called "creep deformations." At high temperatures these deformations are of such magnitude that they definitely influence design. Considerable test data have been obtained at high temperatures, particularly for materials used in the turbine and oil-refining industries. (See, for example, "Compilation of Available High Temperature Creep Characteristics of Metals and Alloys," A. S. M. E.—A. S. T. M., 1938.) Many attempts have been made to interpret these data for the purpose of selecting working stresses.[2] To the structural engineer, the behavior of materials at normal temperatures is of greater interest, but for such temperatures there is little information available. A brief review of the information on creep in simple tension and at normal temperatures will be made first. This study leads to the selection of a new creep-stress law in place of Hooke's law for the development of a mechanics of creep.

Notation

The letter symbols used in this paper are defined where they first appear and are assembled for reference in the Appendix.

Simple Tension; Creep-Stress Law

Usually the simple creep-tension test is used for obtaining basic data on the creep behavior of engineering materials. In this test a specimen is subjected to a stress that remains constant, and the creep deformations are measured with time. With this information creep-time relations can be plotted for each stress value. Fig. 1 shows such plots for Aluminum Alloy 3S-3/4H

[2] "Interpretation and Use of Creep Results," by J. J. Kanter, *Transactions*, A. S. M., Vol. 24, No. 4, December, 1936, p. 870; see also "A Comparison of the Methods Used for Interpreting Creep Test Data," by J. Marin, *Proceedings*, A. S. T. M., Vol. 37, 1937, Pt. II, Technical Papers, p. 258.

tested at room temperature. These experiments in tension and others in pure bending have been reported in detail.[3] For many problems in design where creep is significant, it is necessary to select a working stress based on an allowable creep deformation in the estimated life of the machine or structural member. Industry cannot wait for lifetime test results, and it becomes necessary, therefore, to extrapolate test data that will apply for lengths of time not covered by the test results. This has led to a number of methods for interpreting the test data. For this purpose it is desirable to express a relationship for the creep deformation e_c in a time t for a unit stress s. When elevated temperatures must be considered, there is the additional variable of temperature.

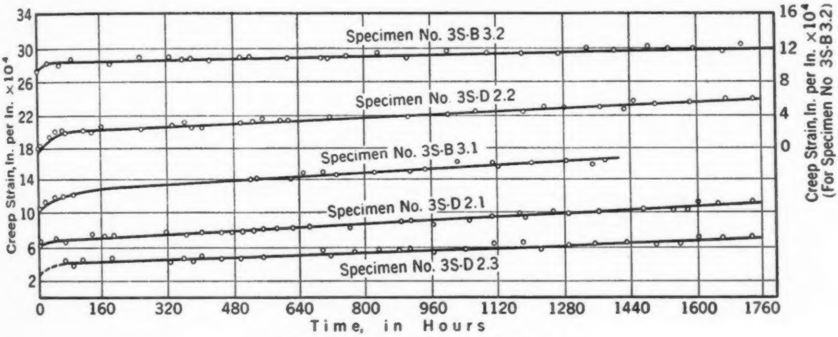

FIG. 1.—TENSION CREEP CURVES

Some of the methods for interpretation developed for elevated temperatures have been presented elsewhere.[2] More recently, procedures for interpretation of data have been proposed by S. H. Weaver,[4] C. R. Soderberg,[5] A. Nádai,[6] and others. A consideration of these methods shows that, although some of the methods fit a particular set of test data better than other methods, they do so by using a more complicated formula. If the creep-stress law is not a simple one, it leads to mathematical difficulties in developing a mechanics of creep, or the expressions deduced become too complicated for practical purposes. A creep law that satisfies the condition of simplicity and has considerable experimental support for many engineering materials at elevated temperatures is the one called the log-log creep-stress relationship. By this method, a constant creep rate is assumed. (See, for example, the many tests reported in "Compilation of Available High Temperature Creep Characteristics of Metals and Alloys," A. S. M. E.—A. S. T. M., 1938. These test results are plotted on a log-log basis and show an approximate linear plot.) In addition, a straight-line relationship is also assumed between the logarithm of the creep rate and the stress—that is,

$$C_n = a\, s^n \qquad (1a)$$

[3] "Creep of Aluminum Subjected to Bending at Normal Temperatures," by J. Marin and L. E. Zwissler, *Proceedings*, A. S. T. M., Vol. 40, 1940, p. 937.

[4] "The Creep Curve and Stability of Steels at Constant Stress and Temperature," by S. H. Weaver, *Transactions*, A. S. M. E., Vol. 58, 1936, p. 745.

[5] "The Interpretation of Creep Tests for Machine Design," by C. R. Soderberg, *ibid.*, p. 733.

[6] "The Influence of Time Upon Creep"—The Hyperbolic Sine Creep Law, Stephen Timoshenko 60th Anniversary Vol., by A. Nádai, Macmillan Co., 1938, pp. 155–171.

in which C_n = constant creep rate $= \frac{e_c}{t}$, and a and n are experimental constants. Eq. 1a also can be written:

$$e_c = a\, s^n\, t \qquad (1b)$$

Considering the behavior of engineering materials at normal temperatures, there are only a few experiments upon which to base a creep-stress law. It is desirable to review briefly some of these tests.

Experiments in tension[3] on Aluminum Alloy 3S-3/4H show that the log-log interpretation expressed by Eqs. 1 is a good approximation. This is shown in Fig. 2 where the logarithm of the constant creep rates obtained from Fig. 1 are plotted versus the logarithm of the stress. The linear relation between these two quantities in Fig. 2 can be expressed by either Eq. 1a or 1b.

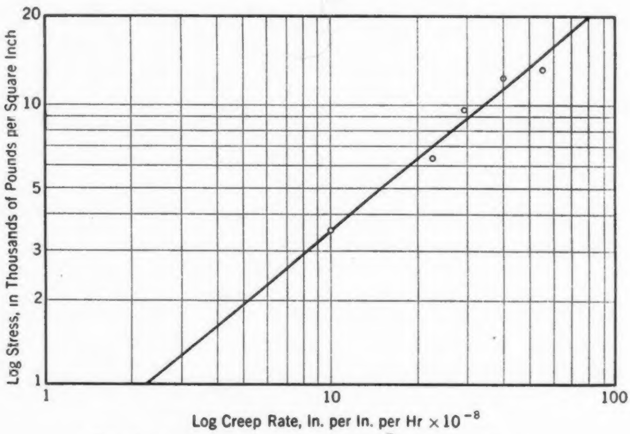

FIG. 2.—RELATION BETWEEN CREEP RATE AND STRESS

Creep tests on various engineering materials at normal temperatures were reported by R. G. Sturm, Assoc. M. Am. Soc. C. E., C. Dumont, and F. M. Howell.[7] These experiments were made on several aluminum alloys, soft and hard copper, and two steels. The material was in the form of sheets and wires. These tests show a log-log plot between the creep deformation and the time. Accordingly, the creep law proposed can be expressed essentially by the equation

$$e_c = F\, t^m \qquad (2)$$

in which F and m are experimental constants. By proposing Eq. 2, no attempt is made to give a general expression involving a variable stress.

Creep experiments on five steels were reported by H. J. French, H. C. Cross, and A. A. Peterson,[8] in which some of the tests were made at room temperature. These experiments were made on:

[7] "A Method of Analyzing Creep Data," by R. G. Sturm, C. Dumont, and F. M. Howell, *Transactions*, A. S. M. E., Vol. 58, 1936, p. A-63.

[8] "Creep in Five Steels at Different Temperatures," by H. J. French, H. C. Cross, and A. A. Peterson, *Paper No. 362*, Technical Papers of the Bureau of Standards, Vol. 22, 1927–1928.

One 0.24% carbon boiler-plate steel specimen;
One structural alloy steel specimen;
One tungsten-vanadium high-speed steel specimen;
One chromium "stain-resisting" steel specimen; and
One nickel chromium "austenitic" steel specimen.

Although sufficient experiments were not made for purposes of establishing a creep-stress law, these tests are significant in giving evidence of creep deformation in structural steel. They show that, for the structural steel tested, creep occurs when the stress is only slightly above the "proportional limit" of the material.

Creep of commercial leads at normal temperatures has been investigated by A. J. Phillips[9] and by H. F. Moore, B. B. Betty, and C. W. Dollins.[10] These experiments indicate that the log-log creep-stress law expressed by Eqs. 1 is a good approximation. Although this material is not of general importance, it is of interest in showing that the log-log plot applies.

Concrete is an important structural material and therefore should also be considered, if possible, in formulating a creep-stress law. Many creep-stress experiments on concrete in compression have been made. A summary of some of the test results and interpretations is given by J. R. Shank,[11] M. Am. Soc. C. E. The recommended creep-stress law on the basis of a study by Mr. Shank,[11] in terms of Eqs. 1 and 2, is

$$e_c = a\, s\, t^m \qquad (3)$$

An examination of the foregoing creep-stress equations shows that a general expression for these equations might be written

$$e_c = a\, s^n\, t^m \qquad (4)$$

Eq. 4 seems to have considerable experimental support and has the advantage of simplicity when compared to some that have been proposed. One disadvantage of most creep-stress relationships, as in the case of Eq. 4, is that the elastic or initial deformation is not included. Referring to the deformation-time relation of Fig. 3, the total unit deformation e at a time t is made up of the elastic or initial deformation e_e plus the creep deformation e_c—that is,

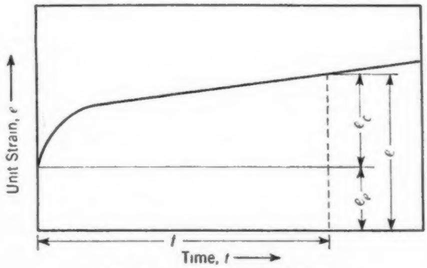

FIG. 3.—CREEP-TIME CURVE

$$e = e_e + e_c \qquad (5a)$$

[9] "Some Creep Tests on Lead and Lead Alloys," by A. J. Phillips, *Proceedings*, A. S. T. M., Vol. 36, 1936, Pt. II, Technical Papers, p. 170.

[10] "The Creep and Fracture of Lead and Lead Alloys," by H. F. Moore, B. B. Betty, and C. W. Dollins, *Bulletin No. 272*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1935; see also "The Investigation of Creep and Fracture of Lead and Lead Alloys for Cable Sheathing," by the same authors, *Bulletin No. 306*, *ibid.*, 1938.

[11] "The Mechanics of Plastic Flow of Concrete," by J. R. Shank, *Proceedings*, A. C. I., 1936, pp. 149–180.

The value of the elastic or initial deformation will be assumed equal to

$$e_e = B\, s^p \quad \text{(5b)}$$

For a material in which Hooke's law applies, $B = 1/E$ and $p = 1$, in which E = the modulus of elasticity of the material. Substituting the values of the deformations from Eqs. 4 and 5b in Eq. 5a, the unit deformation is

$$e = B\, s^p + a\, s^n\, t^m \quad \text{(6a)}$$

Or, for a material for which Hooke's law applies,

$$e = \frac{s}{E} + a\, s^n\, t^m \quad \text{(6b)}$$

Eqs. 6 define the total deformation in terms of the unit stress and the time. In addition, there are five experimental constants. This creep-stress law is used in the next section to develop a theory for stresses and deflections in the case of bending. It is found, however, that this leads to cumbersome expressions, making the formulation of structural mechanics in the case of creep very difficult. For this reason an approximation to Eq. 6a is suggested. This involves assuming the elastic or initial deformation to be expressed by $B\, s^n$ and determining B for the maximum stress—that is, B is determined from the equation $B\, s^2_{max} = s_{max}/E$. The total strain then will be assumed as

$$e = s^n\,(B + a\, t^m) \quad \text{(7a)}$$

An indication of the accuracy obtained in using Eq. 7a is given by an analysis of the test data shown in Fig. 1. Using the results for specimens B3.2 and B3.1, $n = 1.22$, $m = 0.20$, $a = 3.75 \times 10^{-9}$, and $B = 1.04 \times 10^{-8}$. The value of B was determined on the basis of Hooke's law, as explained. The expression for the total strain as given by Eq. 7a then becomes

$$e = s^{1.22}\,(1.04 \times 10^{-8} + 3.75 \times 10^{-9}\, t^{0.20}) \quad \text{(7b)}$$

A comparison of the actual strains with the theoretical values given by Eq. 7b is shown in Table 1. In determining the actual strains, the elastic strains were added to the creep strains. These deformations are for a time t = 1,760 hr. The percentage error in the value of the strains as given by Eq. 7b indicates that for these test data it is a good approximation. The approximation made in obtaining Eq. 7a produces a smaller error than is indicated in the test data. This is because the time covered by the tests is only a small fraction of the estimated life of the structure—that is, the test data cover a period of approximately two months, whereas the limiting deflection would be based on a time on the order of ten or twenty years. The elastic deformation is a very

TABLE 1.—COMPARISON OF ACTUAL AND THEORETICAL STRAINS

(Unit Strains Are Multiplied by 10^5)

Specimen	Unit stress (lb per sq in.)	ACTUAL STRAIN: Creep strain (e_c)	ACTUAL STRAIN: Elastic strain (e_e)	ACTUAL STRAIN: $e_c + e_e$	Theoretical strain (Eq. 7b)	% error
B3.2	3,472	35	29	64	56	−12
D2.3	6,227	70	52	122	115	− 6
D2.1	9,655	90	81	171	163	− 5
D2.2	12,321	110	102	212	271	−28
B3.1	13,130	174	109	283	284	0

small fraction of the creep deformation, and the error produced in using $B\ s^n$ for the elastic strain is therefore very small.

Members Subjected to Bending

For a straight beam subjected to a pure bending moment M, as shown in Fig. 4, a theory for the stress distribution can be developed as in the case of

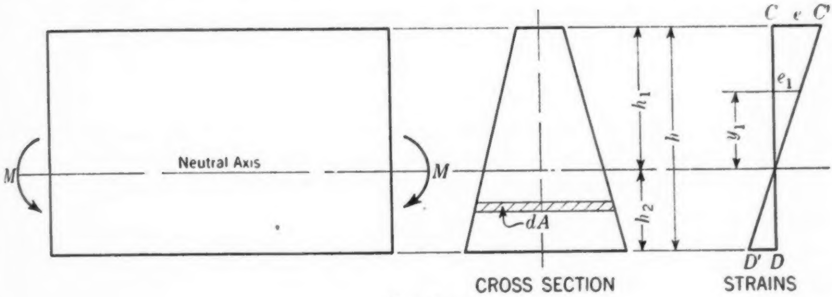

Fig. 4

the elastic theory. To do this it is only necessary to replace Hooke's law in the development, using the elastic theory with the creep-stress law expressed by Eq. 6*a*—that is, the following assumptions will be made:

(1) A transverse cross section of the beam originally plane and normal to the center line of the member remains plane after bending;

(2) For any fiber the strains are defined as in the case of tension by Eq. 6*a*;

(3) The creep-stress relations are identical for fibers in tension and compression at equal distances from the neutral axis; and

(4) The lateral compression between the fibers is negligible.

Assumption (1) has been verified by some experiments in bending.[12] By means of the foregoing assumptions and the conditions of equilibrium, a theory for the stresses and deflections in bending can be obtained. By the creep-stress law of Eq. 6*a*, the strains at distances y_1 and h_1 from the neutral axis are, respectively,

$$e_1 = B\ s^p{}_1 + a\ s^n{}_1\ t^m \quad \text{(8a)}$$

and

$$e = B\ s^p + a\ s^n\ t^m \quad \text{(8b)}$$

Assuming that plane sections remain plane, the strains are proportional to the distances from the neutral axis and

$$\frac{e_1}{e} = \frac{y_1}{h_1} \quad \text{(9)}$$

Placing values of the strains from Eqs. 8 in Eq. 9,

$$y_1 = h_1 \left(\frac{B\ s^p{}_1 + a\ s^n{}_1\ t^m}{B\ s^p + a\ s^n\ t^m} \right) \quad \text{(10)}$$

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[12] "Stresses and Deformations in Pipe Flanges Subjected to Creep at High Temperatures," by J. Marin, *Journal*, Franklin Inst., Vol. 226, November, 1938, p. 645.

For equilibrium the external and resisting moments are equal, or

$$M = \int y_1 \, s_1 \, dA \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

In order to integrate Eq. 11, the position of the neutral axis must be defined. This is given by noting that the summation of stresses over the cross section of the beam must be zero, or

$$\int_{h_1}^{h_2} s_1 \, dA = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Eqs. 10, 11, and 12 completely define the stress in terms of the moment. For a rectangular cross section with width b and depth h, Eq. 12 shows that the neutral axis and centroid coincide. For a rectangular cross section $dA = b \, dy_1$, and Eq. 11 becomes

$$M = b \int y_1 \, s_1 \, dy_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

To evaluate the integral in this equation it is necessary to determine dy_1 from Eq. 10. Placing the values of dy_1 and y_1 in Eq. 13 and integrating, the resulting equation is

$$M = \frac{b \, h^2}{2 \, (B \, s^p + a \, s^n \, t^m)^2}$$

$$\times \left(\frac{p \, B^2 \, s^{2p+1}}{2 \, p + 1} + \frac{2 \, p \, n \, a \, B \, s^{p+n+1} \, t^m}{p + n + 1} + \frac{n \, a^2 \, s^{2n+1} \, t^{2m}}{2 \, n + 1} \right) \ldots\ldots (14)$$

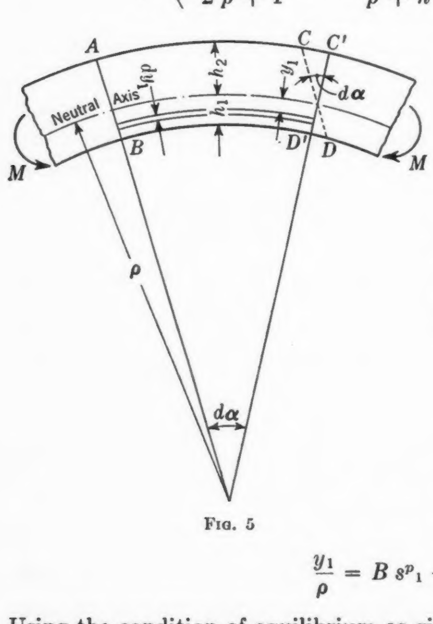


FIG. 5

The stress s on the outer fiber is defined by Eq. 14 in terms of the moment M, dimensions of the beam, and constants of the material.

In order to obtain an expression for the deflection in a beam subjected to pure bending, consider a section of a beam as shown in Fig. 5. The strains at distances y_1 and h_1 from the neutral axis in terms of the curvature are

$$e = \frac{h_1}{\rho} \text{ and } e_1 = \frac{y_1}{\rho} \ldots (15a)$$

in which ρ is the radius of curvature of a curved beam. Placing the values of e_1 from Eq. 8a in Eq. 15a, the curvature is defined by

$$\frac{y_1}{\rho} = B \, s^p{}_1 + a \, s^n{}_1 \, t^m \ldots\ldots\ldots\ldots\ldots\ldots (15b)$$

Using the condition of equilibrium as given by Eq. 13, and placing the values

of y_1 and dy_1 as obtained from Eq. 15*b* in this equation,

$$\frac{1}{\rho} = \frac{2}{h}(B\,s^p + a\,s^n\,t^m) \qquad (15c)$$

For small deflections the relation between the deflection y and the curvature is

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \qquad (15d)$$

The differential equation for the deflection in terms of the stress on the outer fiber is, by Eqs. 15*c* and 15*d*,

$$\frac{d^2y}{dx^2} = \frac{2}{h}(B\,s^p + a\,s^n\,t^m) \qquad (16)$$

A consideration of Eqs. 14 and 16 for the stress and deflection in beams of rectangular cross section shows that these expressions are too complicated for practical purposes. If the value of p is assumed equal to n, the foregoing theory is simplified. This is equivalent to assuming the approximate creep-stress law given by Eq. 7*a*. Placing $p = n$ in Eq. 14, the expression for the maximum stress is

$$s = \left(\frac{M\,h}{I}\right)\left(\frac{2\,n + 1}{6\,n}\right) \qquad (17)$$

in which $I = b\,h^3/12$ = the moment of inertia of the cross section.

The differential equation for the deflection is given by substituting the value of $p = n$ in Eq. 16. Then

$$\left(\frac{D}{B + a\,t^m}\right)\frac{d^2y}{dx^2} = M^n \qquad (18)$$

in which

$$D = \frac{(2\,b)^n\left(\dfrac{h}{2}\right)^{2n+1}}{\left(2 + \dfrac{1}{n}\right)^n} \qquad (19)$$

For the case where $t = 0$ and $n = 1$, Eq. 18 reduces to the equation given by the elastic theory. The value of D can be obtained in a manner similar to the foregoing for symmetrical sections other than rectangular. Eq. 18 can be used for determining deflections in beams in a manner similar to the usual double integration method. It is only necessary to integrate Eq. 18 twice and to obtain the constants of integration by using the boundary conditions of the particular problem. For example, in a simply supported beam of length l, subjected to a center concentrated load P, the deflection at a distance x from one end is

$$y = \frac{B + a\,t^m}{D}\left[\frac{x^{n+2}}{(n+1)(n+2)} - \frac{\left(\dfrac{l}{2}\right)^{n+1}}{(n+1)}\,x\right]\frac{P^n}{2^n} \qquad (20)$$

For a value of $x = l/2$, the maximum deflection is, by inspection or by Eq. 20,

$$y_{\max} = \frac{B + a\,t^m}{D} \times \frac{l^{n+2}}{2^{2(n+1)}} \times \frac{P^n}{n+2} \quad \ldots\ldots\ldots\ldots\ldots (21)$$

The accuracy of the foregoing theory is indicated by a comparison between the measured and theoretical deflections in aluminum alloy beams subjected to pure bending.[3] These beams were subjected to pure bending moments of values shown in Table 2. They were of rectangular cross section of width $b = 0.50$ in. and depth $h = 1.00$ in., and the deflections were measured over a gage length $l = 8$ in. By Eq. 18 the expression for the deflection becomes

$$y = \frac{B + a\,t^m}{D} \times \frac{l^2}{8} \times M^n. \quad (22)$$

TABLE 2.—COMPARISON OF EXPERIMENTAL AND THEORETICAL DEFLECTIONS

Specimen No.	Moment (in-lb)	Theoretical deflection (in.)	Actual deflection (in.)	% error
D5	687	0.018	0.013	28
A3	996	0.029	0.022	24
C3	1,333	0.040	0.031	22
A4	1,667	0.053	0.046	13

Using the values for the experimental constants obtained from the tension tests of Fig. 1, the theoretical values of the deflections, as given by Eq. 22, are shown in Table 2. The experimental values of these deflections are also listed. A comparison between the theoretical and experimental values shows that the theory gives a good approximation. In making this comparison the preliminary nature of these tests in this new field of study must be considered.

STATICALLY INDETERMINATE STRUCTURES

Knowing the relationship between the stresses and deformations in the case of simple tension and pure bending, theories for the analysis of statically indeterminate structures can be developed. It is convenient to discuss first statically indeterminate beams of one span.

(*1*) *Double Integration Method—Beams of One Span.*—For beams that are statically indeterminate, the values of the reactions and deflections can be found in a manner similar to the double integration method used for the elastic case—that is, essentially it is only necessary to integrate the differential Eq. 18 twice, first substituting the value of the moment M in terms of x for the particular problem. The constants of integration and the statically indeterminate reactions are then found by using the particular boundary conditions of the problem. The difficulties in this procedure are mathematical because in some cases the integration is not direct and in others considerable work is involved. These difficulties arise since the moment M in Eq. 18 is raised to a power n that is usually different from one, and the simplifications afforded in the elastic case by using the methods of superposition can no longer be used in this analysis. Some examples of statically indeterminate beams will now be discussed and the maximum moments compared with the "elastic" values.

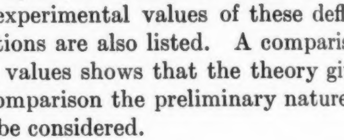


FIG. 6

Example 1.—A beam fixed at one end, simply supported at the other, and subjected to an end moment m is shown in Fig. 6. By Eq. 18 the deflection equation is

$$C_n \frac{d^2y}{dx^2} = M^n = (R_A\, x - m)^n \qquad (23)$$

in which R_A = the reaction at point A, and

$$C_n = \left(\frac{D}{B + a\, t^m} \right) \qquad (24)$$

Integrating Eq. 23 twice,

$$C_n \frac{dy}{dx} = \frac{(R_A\, x - m)^{n+1}}{R_A\,(n+1)} + c_1 \qquad (25a)$$

and

$$C_n\, y = \frac{(R_A\, x - m)^{n+2}}{R^2_A\,(n+1)\,(n+2)} + c_1\, x + c_2 \qquad (25b)$$

in which c_1 and c_2 are constants of integration. Using the boundary conditions that the deflections are zero at both ends of the beam and the slope is zero at point B, the value of the reaction R_A is determined by the equation

$$R_A\, l\,(n+2)\,(R_A\, l - m)^{n+1} - (R_A\, l - m)^{n+2} + (-\, m)^{n+2} = 0 \qquad (26a)$$

Noting that $R_A\, l - m = M_B$, Eq. 26*a* can be written in terms of M_B—that is,

$$M^{n+2}{}_B + \left(\frac{n+2}{n+1} \right) m\, M^{n+1}{}_B + \left(\frac{1}{n+1} \right) (-\, m)^{n+2} = 0 \qquad (26b)$$

The value of the moment M_B as defined by Eq. 26*b* will depend upon the value of n. For different values of n the variation in M_B is represented in Table 3. The table also shows that there is a 44% increase in the moment at B for $n = 7$ over the value determined by the elastic theory. Values of n at elevated temperatures for steel have been reported as high as 9 so that it is possible to have an appreciable increase in the value of the moments over those values obtained by elasticity. The value of the bending stress, as determined by Eq. 17 for a rectangular section, is also modified, depending upon the value of n. In estimating the resultant effect it is necessary, therefore, to consider the influence of the change in moments as well as the change in stress distribution. Another important consideration is that of limiting the maximum deflection obtained from Eq. 25*b*.

TABLE 3.—VALUES OF M_B

Value of n	M_B	% increase
1 (elastic case)	0.5 m	0
3	0.65 m	30
5	0.67 m	34
7	0.72 m	44

Example 2.—A uniformly loaded beam, fixed at one end and simply supported at the other, is shown in Fig. 7. The differential equation for this problem, using Eq. 18, is

$$C_n \frac{d^2y}{dx^2} = \left(R_A\, x - \frac{w\, x^2}{2} \right)^n \qquad (27)$$

in which w is the load per unit of length. Proceeding in a manner similar to Example 1, an equation for the reaction R_A at point A can be determined. For a value of $n = 3$ the equation for R_A is

$$R^3_A - (1.25\, w\, l)\, R^2_A + (0.535\, w^2\, l^2)\, R_A - 0.078\, w^3\, l^3 = 0 \qquad (28)$$

From Eq. 28 the value of R_A is $0.40\, w\, l$. For the elastic case, $R_A = 0.375\, w\, l$. The moment at the support for $n = 3$ is, by statics, $M_B = 0.10\, w\, l^2$ as compared to $0.125\, w\, l^2$ by the elastic case.

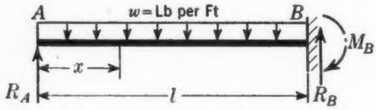


FIG. 7

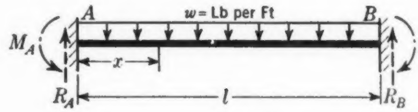


FIG. 8

Example 3.—For a beam fixed at both ends and subjected to a uniform load, as shown in Fig. 8, Eq. 18 becomes

$$C_n \frac{d^2y}{dx^2} = \left(\frac{w\, l\, x}{2} - M_A - \frac{w\, x^2}{2} \right)^n \qquad (29)$$

Again following the procedure as explained for Example 1—namely, integrating Eq. 29 twice and applying the boundary conditions—an equation defining the value of M_A can be determined. For $n = 3$ this relation is

$$M^3_A - (0.25\, w\, l^2)\, M^2_A + (0.025\, w^2\, l^4)\, M_A - 0.00088\, w^3\, l^6 = 0 \qquad (30)$$

Solving for this moment, its value is $M_A = 0.074\, w\, l^2$ as compared with $M_A = 0.0833\, w\, l^2$ for the elastic case.

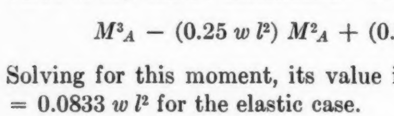


FIG. 9

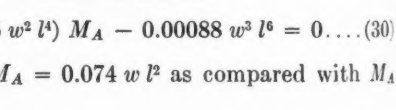


FIG. 10

Example 4.—For a beam fixed at both ends and subjected to a concentrated load (Fig. 9), it is necessary to write equations for both parts AB and BC of the beam. This leads to equations that are too complicated to present in this paper. If the load is considered at the center, $R_A = P/2$ and Eq. 18 need be written for only one half the beam. Thus, for AB,

$$C_n \frac{d^2y}{dx^2} = \left(\frac{P}{2}\, x - M_A \right)^n \qquad (31)$$

Integrating this equation twice and applying the necessary boundary conditions, an equation for M_A can be found—namely,

$$\left(\frac{P\, l}{4} - M_A \right)^{n+1} - (-\, M_A)^{n+1} = 0 \qquad (32)$$

Using this relation, $M_A = 0.13\,P\,l$ for $n = 3$, and $M_A = 0.14\,P\,l$ for $n = 5$. These values are higher than that of $0.125\,P\,l$ as found for the elastic case.

The foregoing examples show that the values of the statically indeterminate reactions and moments may be appreciably different from the magnitudes obtained for the elastic case. It may be necessary, therefore, to consider this difference in the design of statically indeterminate beams in which creep occurs.

(*2*) *Slope-Deflection Method.*—The slope-deflection method is one of the classical methods for structural analysis. For the condition of creep, equations similar to those for the elastic case can be derived. To obtain these equations, consider the beam AB in Fig. 10 simply supported at points A and B and subjected to moments M_1 and M_2 at the ends. By the slope-deflection method these moments are determined in terms of the slopes or rotations θ_1 and θ_2 at the ends of the beam. To do this, Eq. 18 can be used. Then

$$C_n \frac{d^2y}{dx^2} = M^n = (R_1\,x - M_1)^n \qquad (33)$$

Integrating this equation twice,

$$C_n \frac{dy}{dx} = \frac{(R_1\,x - M_1)^{n+1}}{R_1\,(n+1)} + c_1 \qquad (34a)$$

and

$$C_n\,y = \frac{(R_1\,x - M_1)^{n+2}}{R^2{}_1\,(n+1)\,(n+2)} + c_1\,x + c_2 \qquad (34b)$$

Using the conditions that the deflections are zero at both $x = 0$ and $x = l$, the constants of integration c_1 and c_2 are determined. Then the slopes θ_1 and θ_2 are the values of dy/dx at $x = 0$ and $x = l$, respectively, as found by Eq. 34*a*. Noting also that $R_1 = (M_1 + M_2)/l$, the slope-deflection equations expressing the slopes in terms of the moments are:

$$C_n\,\theta_1\,(M_1 + M_2)^2\,(n+2)\,(n+1) = (M_1 + M_2)\,(-\,M_1)^{n+1}\,l\,(n+2) + l\,(-\,M_1)^{n+2} - (M_2)^{n+2}\,l \qquad (35a)$$

and

$$C_n\,\theta_2\,(M_1 + M_2)^2\,(n+2)\,(n+1) = (M_1 + M_2)\,(M_2)^{n+1}\,l\,(n+2) + l\,(-\,M_1)^{n+2} - (M_2)^{n+2}\,l \qquad (35b)$$

The solution of Eqs. 35 simultaneously for M_1 and M_2 offers difficulties not present in the elastic case. The conclusion must be, therefore, that the slope-deflection method is not suitable and that the modified moment-distribution method should be used.

(*3*) *Moment-Distribution Method.*—A study of the moment-distribution method of structural analysis as applied to members subjected to creep shows again the fundamental nature of the Hardy Cross method.[13] The basic character of this classic method remains unchanged for applications where creep must be considered, but it is necessary to determine the beam constants—namely, the fixed-end moments, the carry-over factors, and the distribution factors.

[13] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

The determination of the fixed-end moments has been explained herein in the section on the double integration method. To determine the carry-over and distribution factors it is necessary to determine the slope at the left end of the beam in Example 1, as shown in Fig. 6. This slope is the value of dy/dx in Eq. 23 for the value $x = 0$ or

$$C_n \theta_A = m^n \left\{ \frac{l\left[(-1)^{n+1} - (k)^{n+1}\right]}{(k+1)(n+1)} \right\} \qquad (36)$$

or

$$m^n = \frac{C_n (k+1)(n+1)}{l\left[(-1)^{n+1} - (k)^{n+1}\right]} \theta_A = J\,\theta_A \qquad (37)$$

in which

$$k = \frac{M_B}{m} \qquad (38a)$$

and

$$J = \frac{C_n (k+1)(n+1)}{l\left[(-1)^{n+1} - (k)^{n+1}\right]} \qquad (38b)$$

Eq. 37 expresses the moment m at A in terms of the slope θ_A at A. If the arrangement of members shown in Fig. 11 is considered, then, by Eq. 37,

$$M^n_{ab} = J_{ab}\theta_a; \quad M^n_{ac} = J_{ac}\theta_a; \quad M^n_{ad} = J_{ad}\theta_a; \quad \text{and} \quad M^n_{ae} = J_{ae}\theta_a \qquad (39)$$

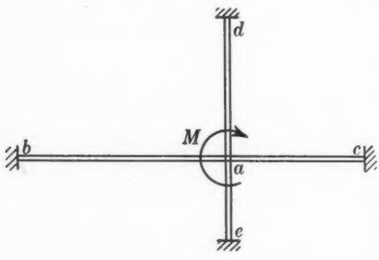


Fig. 11

For equilibrium,

$$M_{ab} + M_{ac} + M_{ad} + M_{ae} + M = 0 \qquad (40)$$

Substituting values from Eq. 39 in Eq. 40, and solving for θ_a,

$$\theta_a = -\frac{M^n}{(\Sigma J^{1/n})^n} \qquad (41)$$

Placing this value of the slope θ_a in Eq. 39, the ratios of the moments in the members to the applied moments—called the distribution factors—are:

$$\frac{M_{ab}}{M} = \frac{J^{1/n}{}_{ab}}{\Sigma J^{1/n}}; \quad \frac{M_{ac}}{M} = \frac{J^{1/n}{}_{ac}}{\Sigma J^{1/n}}; \quad \frac{M_{ad}}{M} = \frac{J^{1/n}{}_{ad}}{\Sigma J^{1/n}}; \quad \text{and} \quad \frac{M_{ae}}{M} = \frac{J^{1/n}{}_{ae}}{\Sigma J^{1/n}} \qquad (42)$$

or

$$D_f = \frac{J^{1/n}{}_p}{\Sigma J^{1/n}} \qquad (43)$$

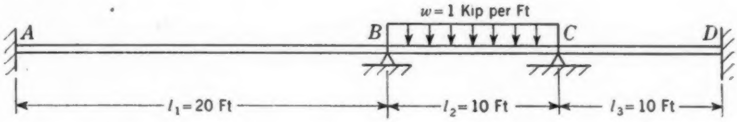
in which D_f is the distribution factor for any member.

The carry-over factor can be obtained by reference to Fig. 10 and Eq. 35b, which give the slope at point B in the beam AB. If this slope, $\theta_2 = \theta_B$, is placed equal to zero, the moment M_2 is then the moment at the fixed end of a beam AB, fixed at point B, simply supported at A, and subjected to a moment M_1 at A. The ratio M_2/M_1 is then the carry-over factor.

By Eqs. 35b, with $\theta_2 = 0$, the carry-over factor, $C_f = M_2/M_1$, is defined by the equation:

$$C^{n+2}{}_f + \left(\frac{n+2}{n+1}\right) C^{n+1}{}_f - \left(\frac{1}{n+1}\right) = 0 \qquad (44)$$

For values of n from 1 to 7 the carry-over factor varies from 0.5 to 0.72. With the fixed-end moments, distribution factors, and carry-over factors determined, the solution of some particular problem can be found, as illustrated by the following example.



(a) Computation for Moments at Supports

C_f	0.61		0.61		0.61	
D_f		0.44	0.56	0.50	0.50	
Fixed-End Moments			−7.40	−7.40		
		−3.26	+4.14	+3.70	−3.70	
	+1.98		−2.26	−2.53		+2.26
		−1.00	+1.26	+1.26	−1.26	
	+0.61		−0.77	−0.77		+0.77
		−0.34	+0.43	+0.38	−0.38	
	+0.21		−0.23	−0.26		+0.23
		−1.0	+0.13	+0.13	−0.13	
Moments at Supports	+2.80	−4.70	−4.70	−5.47	−5.47	+3.26

(b) Comparative Computations for the Elastic Case

C_f	0.5		0.5		0.5	
D_f		0.33	0.67	0.50	0.50	
Fixed-End Moments			−8.83	−8.83		
		−2.78	+5.55	+4.16	−4.16	
	+1.39		−2.08	−2.78		+2.08
		−0.68	+1.40	+1.39	−1.39	
	+0.34		−0.70	−0.70		+0.70
		−0.23	+0.47	+0.35	−0.35	
	+0.12		−0.18	−0.24		+0.18
		−0.06	+0.12	+0.12	−0.12	
Moments at Supports	+1.85	−3.75	−3.75	−6.02	−6.02	+2.96

FIG. 12.—CONTINUOUS BEAM OF THREE SPANS

Example 5.—A continuous beam of three spans is shown in Fig. 12. The moments at the supports will be determined by the moment-distribution method for a value of $n = 3$, and these values will be compared with those for the elastic case.

In order to determine the moments at the supports, the beam constants must be obtained. The fixed-end moments for a uniform load and a value of

$n = 3$ are determined in Example 3. These moments are $M = 0.074\, w\, l^2 = 0.074 \times 1 \times 100 = 7.40$ kip-ft. The carry-over factor for a value of $n = 3$ is determined by Eq. 44. Its magnitude is $C_f = 0.61$. Finally, the distribution factors are found by Eq. 43, using Eq. 38*b* to determine J_p, and Eq. 38*a* to find k. In Example 1 the value of $k = 0.65$ is given for a value of $n = 3$. By Eq. 38*b*, $J_{ab} = 8.02\ (C_n/l) = (8.02/l_{ab})\ C_n$. Similarly, $J_{bc} = 8.02\ (C_n/l_{bc})$.

The distribution factor for member AB is now calculated by Eq. 43—namely,

$$D_{ab} = \frac{(J_p)^{1/n}}{\Sigma(J^{1/n})} = \frac{J^{1/n}{}_{ab}}{J^{1/n}{}_{ab} + J^{1/n}{}_{bc}} = \frac{(1/l_{ab})^{1/n}}{(1/l_{ab})^{1/n} + (1/l_{bc})^{1/n}} \quad \text{......} (45)$$

or $D_{ab} = 0.44$; and similarly, $D_{bc} = 0.56$. Using the beam constants as given in Fig. 12(*a*), the moments at the supports are found. In Fig. 12(*b*) these moments are determined for the elastic case for the purpose of comparison. A comparison of these moments indicates that there is sufficient difference in the moments to warrant consideration in design. Other types of statically indeterminate structures can be analyzed in a similar manner.

A consideration of the energy methods of structural analysis shows that these methods cannot be applied in the case of creep because they require that the material be elastic. Other fundamental methods of structural analysis used in the elastic theory do not apply in the case of creep as, for example, the use of the law of superposition and the determination of influence lines by mechanical means. On the other hand, secondary stresses in the case of creep can be determined since the moment-distribution method can be applied.

Conclusion

In this paper a new creep-stress law is proposed for materials in which creep at both normal and elevated temperatures is important. Some experimental evidence is given and other data are cited that show the validity of the creep law assumed. A theory for stresses and deflections in members subjected to bending is developed and the values compared with creep test results. Theories are then derived for the analysis of statically indeterminate structures in cases where creep is present. Comparison of results with the elastic case shows that there may be an appreciable difference in the values of reactions and moments. It should be noted that there is considerable need for experimental work on this problem. In the meantime, however, this paper shows, in general, how structural analysis is modified when creep occurs in engineering materials.

APPENDIX

Notation

The following letter symbols, used in this paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:[14]

[14] ASA—Z10a—1932.

A = area; dA = element of area;
a = experimental constant; also B, F, m, n, and p;
B = (see a)
b = breadth;
C_n = constant creep rate: C_f = carry-over factor;
c_1 and c_2 = constants of integration;
D = a dimension constant: D_f = distribution factor;
E = modulus of elasticity;
e = strain, unit deformation;
F = (see a);
h = height or depth of beam:
h_1 = distance from neutral axis to top of beam;
h_2 = distance from neutral axis to bottom of beam;
I = moment of inertia;
J = (see Eq. 38b): J_p, J_{ab}, etc., refer to given members of a structural arrangement;
k = a substitution constant (Eq. 38a);
l = span length: l_1, l_2, etc., denote lengths of adjacent spans 1, 2, etc.;
M = bending moment, subscripts denoting points about which moments are taken;
m = (see a);
n = (see a);
P = concentrated load;
p = (see a);
R = reaction at the point denoted by subscript;
s = unit stress;
t = time;
w = uniformly distributed load;
x = coordinate distance from the left support;
y = ordinate to an elastic curve; deflection of a beam;
α = angle of curvature between two points on a curved beam;
θ = slope of the elastic curve at points designated by subscripts; and
ρ = radius of curvature of beam.

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PAPERS

RELATIVE ANGULAR, LINEAR, AND TRAVERSE ACCURACIES IN CITY SURVEYS

By S. A. Bauer,[1] Assoc. M. Am. Soc. C. E.

SYNOPSIS

The methods, procedures, and accuracies of the private practicing surveyor in the past were determined by the algebraic sum of a number of purely local and personal conditions. Each individual locality had its own standards and each surveyor within the locality was an authority unto himself.

Standardization of title search and mortgaging practices throughout the United States now combines with an awakened professional consciousness among the surveyors to demand a standardization of procedure, accuracy, and practice. Much is being accomplished on the standards of practice, but on the question of accuracy (compatible with the peculiarities of the profession) there is little information.

Existing investigations into survey accuracy fall generally into (1) studies of surveys of high precision for specialized purposes, and (2) studies of purely instrumental accuracies under carefully controlled conditions. Although both are interesting, neither indicates accuracies the surveyor can depend on under the conditions and with the equipment that the economics of the profession force upon him.

This investigation was made in that direction. The study is made of actual work that was performed over a period of fifteen years, for clients, in the regular private practice of surveying. The methods and equipment that have been used by the writer for that period are indicated merely for record and are not held out as necessarily good or advisable.

Accuracies as obtained in active practice (angular, linear, and traverse) are tabulated. The factors contributing to accuracy are investigated and certain conclusions that seem to be indicated by the facts, relative to methods and

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by October 1, 1942.

[1] (Bauer Surveys Co.), Cleveland, Ohio.

equipment, are advanced. It is the hope of the writer that similar studies will be made in other localities with other methods and equipment so that a true standard of obtainable accuracy can be determined for the practicing surveyor.

Introduction

The purpose of the study was to determine the accuracies that were obtained in survey practice by the office of the writer, a surveyor in the metropolitan area of Cleveland, Ohio, with the ordinary equipment of the profession, under conditions imposed by business necessity, traffic and weather, and economic considerations. The equipment used was: 1-min transits of standard American manufacture; 100-ft steel tapes (graduated throughout into feet, tenths, and hundredths), $\frac{1}{4}$ in. wide and 0.015 in. thick; plumb bobs; hand levels; thermometers graduated to 2° F; sighting rods (6-ft and 8-ft lengths of $\frac{1}{2}$-in. round solid steel); and spring balances graduated in half-pound intervals from 0 to 30 lb.

Observation Methods

Angular observations were made in the usual manner. The instrument was set carefully over the point by plumb bob, and stations being observed were marked by sighting rods, which were generally plumbed by balancing in the hands of the rodman and checked by instrument. If the day was windy, or if only a small part of the rod was visible to the instrumentman, the rodman plumbed his rod by plumb bob. When a rod was "planted" instead of being held during the observation, the "planting" was done by instrument.

The angle was generally observed in a clockwise direction, sighting from the center of one rod to the center of the other. All angles were turned six times for a set. The first and sixth turns were read and recorded to the nearest 30″ by estimation. The final reading was divided by six for the value of the set and angle. Only one vernier was read for economy of time. If the mean of six turns was more than 30″ removed from the first reading, the angle was re-observed. In most cases, all six turns were made with the telescope in one position. Where a re-observation was made, the telescope was inverted. (Subsequently this procedure was changed to three turns with telescope direct and three reversed.)

In taping, the tape was generally held suspended throughout. The difference in level between tape ends was observed by hand level and added to the amount necessary to clear obstructions. Points were transferred to the ground by plumb bob.

Tape lengths were marked by arrows drawn with lumber crayon when the taping was over pavement or sidewalk. In other cases the ground was leveled with the foot and a small headed nail was used to mark the point.

Only in recent years has a spring balance been used. About 80% of the measurements under study herein were made without the use of a spring balance. In those cases the tension was estimated with a tendency toward overpull in order to compensate for taping errors which in general have the effect

of a short tape. The tape was lined in by eye by the rear tapeman, the line ahead being marked by a sighting rod. In all cases, after setting a point for a tape end mark, a check measurement was made before proceeding.

It is not the intent of this paper to advocate the foregoing procedure as being either good or proper. The writer merely catalogs the methods and equipment that produced the data herein studied.

Extent of Tabulations and Working Conditions

Since the purpose of this study was to determine accuracies actually obtained in practice, no lines, angles, or traverses were run particularly for this study. The observations used herein represent all field notes and calculations, of the period from 1925 to 1940, applying to specific jobs that this office was hired to perform.

The work included in this study is almost exclusively confined to metropolitan Cleveland. About 60% of the work is within the areas of city development and about 40% in the outlying country. About 75% of the city work is in flat territory with grades that rarely exceed 4%, and the remaining 25% is in territories where the grades generally vary between 8% and 12%.

The outlying country work, involving about 40% of the work studied herein, is distributed as follows: 40% in flat country with grades up to 5%; 20% in rolling country with grades between 5% and 10%; the remaining 40% in rough country ranging from moderately rough terrain to extremely rough where distances had to be measured by slope measurements and vertical angle.

The lengths of traverse sides and of other measured lines vary from 50 ft to 4,000 ft with a mean length of 700 ft for all lines. Traverses vary in total length from 500 ft to 18,000 ft and from 4 to 37 sides per traverse. The average length of traverse is 4,300 ft.

The work studied herein was performed during all months of the year. All climatic conditions are represented except heavy rain or snow, the temperature ranging from 15° F to 95° F.

Methods Used in This Study

To compare angular and linear accuracies and the effect of these accuracies on traverse closures, a common unit for their measurement was determined; that is, the error was expressed as one part in x thousands. This quantity was obtained in linear observations by dividing the calculated probable error of a single measurement into the length of the line. In traverse closures it was obtained by dividing the error of closure into the length of the traverse.

In angular observations the probable error of each single set of six turns was calculated in terms of seconds of angle. The tangents of the angular errors converted to one part per x thousands were then determined and used as the expression indicating the linear effect of the angular error.

In most of the graphs which accompany this study, the vertical coordinates represent the accuracies in terms of one part in x thousands; the horizontal coordinates represent the percentages of the various accuracies. By a reciprocal method, the accuracies in terms of one part in x thousands can also

be expressed as y parts per hundred. This latter expression of accuracy is hereinafter used for purposes of establishing mean values.

The probable error of a single observation is expressed by the formula

$$r_s = 0.6745 \sqrt{\frac{\Sigma v^2}{N-1}} \qquad (1)$$

in which r_s is the probable error of a single observation, v is the residual from the mean, and N is the number of observations. When $N = 2$, Eq. 1 becomes,

$$r_s = 0.9539\, v \qquad (2)$$

Further, the probable error of the mean of two or more observations is expressed by the following formula:

$$r_m = 0.6745 \sqrt{\frac{\Sigma v^2}{N\,(N-1)}} \qquad (3)$$

which, when $N = 2$, becomes

$$r_m = 0.6745\, v \qquad (4)$$

Tabulation of Angular Readings

To determine the accuracy of a single reading of a 1-min transit, 176 sets of angles were tabulated. In each case the first reading and the mean of six turns were recorded, together with the difference between the first reading and the mean of six. These results were as follows:

Difference, in seconds, between the first reading and the mean of six (equal to, or less than)	Percentage of total (176) readings
0	15
10	41
20	63
30	83
40	89
50	96
60	98
70	100

The mean difference of the entire 176 angles tabulated was found to be 21″.

Sixty-six angles re-observed from the same instrument setup were then tabulated. The probable error of each single set (the mean of six turns) of each pair of sets was calculated and tabulated and the percentages of the various accuracies determined. The results are plotted as curve 1 in Fig. 1. The mean probable error of the 66 single angles studied in this classification is 3.8″. It is to be noted that all of the angles represented in this classification were re-observed from the same instrument setup, with no appreciable time lapse between observations.

In Fig. 1, the mean value is plotted as a short horizontal line intersecting curve 1 at 3.8″. In order that later in this study the angular and linear accuracies may be compared, curve 1 was then converted into curve 3, shown

in Fig. 2, by plotting the tangents (expressed in terms of one part in x thousands) of all angular values of curve 1, thereby giving an angular-error curve in the common unit of one part in x thousands.

After a time lapse ranging from a few days to ten years, 70 angles were then re-observed and studied. Again, the probable error of each single set (the mean of six turns) of each pair of sets was calculated and tabulated and the

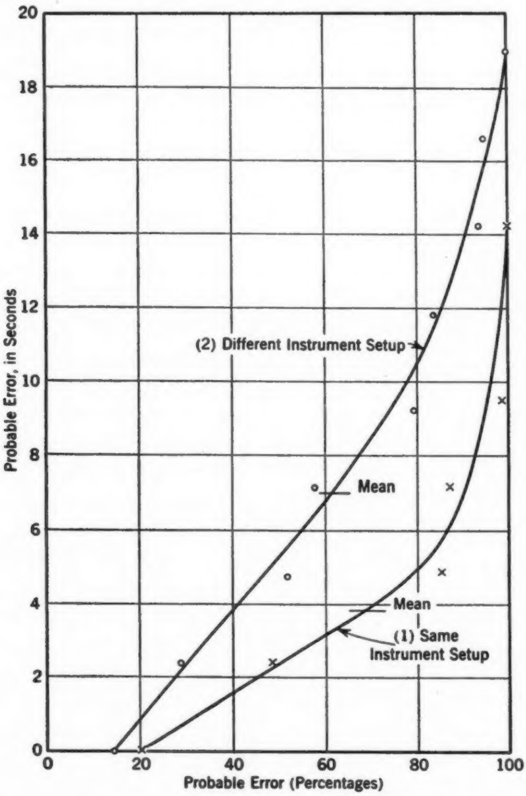


FIG 1.—ANGULAR ACCURACIES IN SECONDS OF ARC; PROBABLE ERRORS OF SINGLE SETS OF SIX TURNS

percentages of the various accuracies determined. The results are plotted as curve 2 in Fig. 1. The mean probable error of the 70 single angles studied in this classification is 7.0″. The mean value was plotted as a short horizontal line intersecting curve 2 at 7.0″ (see Fig. 1).

Again, in order to compare angular and linear accuracies, curve 2, Fig. 1, was converted into curve 4, Fig. 2, by plotting the tangents (expressed in terms of one part per x thousands) of all angular values of curve 2, giving this angular-error curve in the common unit of one part in x thousands.

Tabulations of Linear Readings

For this study 120 lines, retaped by the same party within the same day, were tabulated. The probable error of each single measurement of the pairs of measurements was calculated and converted into the unit of one part per x thousands. The mean of the probable errors and the percentages of the various accuracies were determined. The mean probable error of these 120 line measurements was found to be 0.00319 parts per hundred, or one part in 31,300. These values were plotted as curve 5, Fig. 3(a). The mean value was plotted as a short horizontal line intersecting curve 5 at 31,300.

Next, 274 lines, remeasured during the 15-yr period, but with time lapses between the two measurements varying from a few days to twelve years, were tabulated. As before, the probable error of each single measurement was calculated and translated into the unit used herein; the accuracies were segregated into percentages, and the mean of the probable errors was determined. The mean probable error of these 274 measurements was found to be 0.00439 part per hundred, or one part per 22,800. These values were plotted as curve 6, Fig. 3(a). Again the mean value is indicated as a short horizontal line intersecting curve 6 at 22,800.

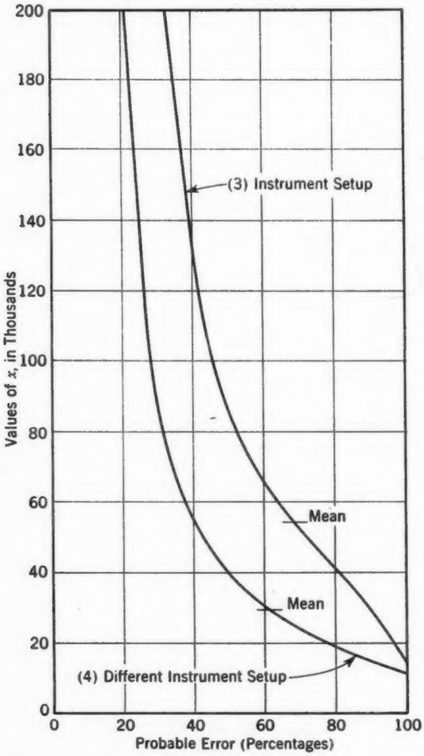


Fig. 2.—Angular Accuracies Expressed as One Part in x Thousands

Tabulations of Traverse Notes

During the 15-yr period, 92 traverses were run and tabulated by their errors of closure expressed in terms of one part in x thousands. (Due to lack of a precise control in this territory, all traverses herein are closed upon themselves as a complete loop.) The percentage of the various accuracies was determined as in previous investigations of angular and linear accuracies, and the mean of the traverse errors was determined. The mean traverse closure error of the 92 traverses under consideration was found to be 0.00442 part per hundred or one part in 22,600. These values were plotted as curve 8, Fig. 4, the value of the mean error again being indicated by a short horizontal line at 22,600.

Findings of Fact from Tabulations

To analyze more clearly the facts determined by this investigation, curves 4 and 6 are plotted in Fig. 4, with curve 8. These are the accuracy curves of

angles observed at different times, lines measured at different times, and traverse closures. The mean value of each curve is also indicated in the manner previously described, as well as the levels of first-order, second-order, and third-

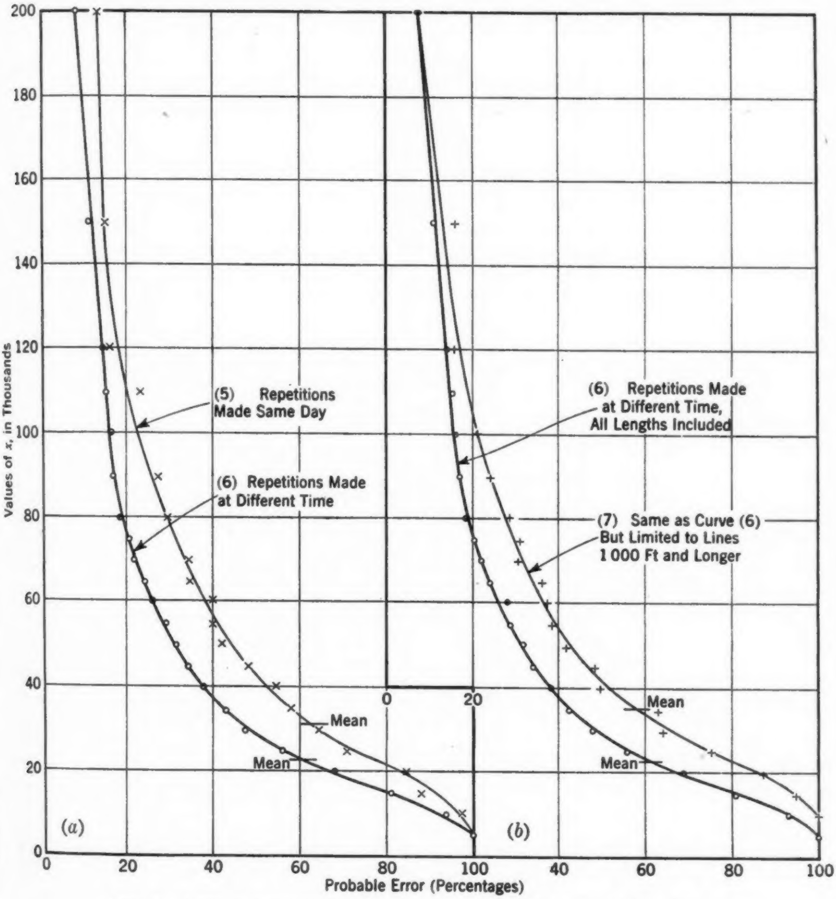


FIG. 3.—LINEAR ACCURACIES, EXPRESSED AS ONE PART IN x THOUSANDS

order accuracies, according to the Society's *Manual of Engineering Practice No. 10* ("Technical Procedure for City Surveys"). Examination of the curves in Fig. 4 and the data from which they were plotted discloses the following facts:

(*a*) All traverses run in the 15-yr period close with a third-order accuracy or better;

(*b*) 83% of the traverses close with a second-order accuracy or better;

(*c*) 39% of the traverses close with a first-order accuracy or better;

(*d*) The mean of angular accuracy exceeds the mean of linear accuracy by approximately 30%;

(*e*) The curve of traverse closure accuracy is practically identical with the curve of linear accuracy in the normal ranges;

(*f*) The probable error in a single reading of a 1-min transit is $\pm 25''$. The mean difference between a single reading of a 1-min transit in a group of six readings, and the mean of that group is $\pm 21''$;

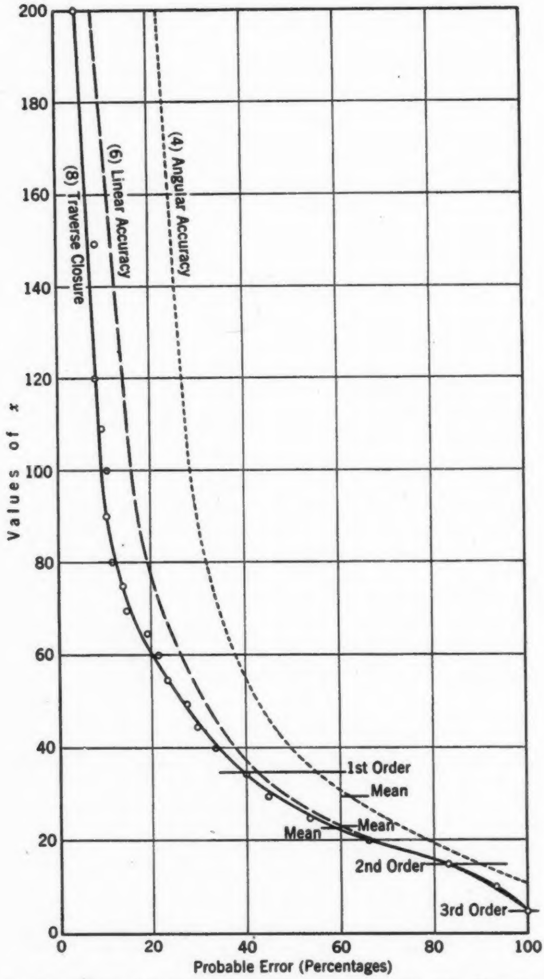


FIG. 4.—TRAVERSE, MEASUREMENT, AND ANGLE ACCURACIES, EXPRESSED AS ONE PART IN x THOUSANDS

(*g*) The mean probable error of an angle re-observed after an appreciable time interval with a 1-min transit turned six times is $7''$, or a mean difference of $15''$ between the two observations;

(*h*) The mean accuracy of angles re-observed at the same time from the same instrument setup is 84% greater than the mean accuracy of angles re-observed after an appreciable time lapse; and

(*i*) The mean accuracy of lines double taped at the same time is 37%

greater than the mean accuracy of lines remeasured after an appreciable time lapse.

Factors of Accuracy

The foregoing data provide information on the basis of which it is possible to study some of the causes of inaccuracies under actual working conditions with a view toward determining the limits of accuracy and the extent to which accuracy can be increased within the limitations imposed by the economics of the profession, traffic, weather, and local conditions. For example, the variations in the accuracies of angles observed from the same setup can be compared with those of angles observed at different times—a difference of 84%, as indicated previously.

The factors that differ in the two categories which might account for the variation are:

(1) Variations in instrument adjustment, principally in the line of collimation;
(2) Movement of points;
(3) Errors in centering the transit over the point;
(4) Errors in the marking of observed stations; and
(5) Refraction of light due to atmospheric conditions.

It is to be noted that, except for the movement of points, all factors tend to indicate the lower accuracy (the one obtained by re-observations of different times) as the more probably correct one.

Item (1), the adjustment of the instrument, would seem to be of little significance in the variation of the accuracies, since in inverting the telescope for the re-observation of angles from the same setup, the collimation error is corrected in much the same degree that it is by varying adjustments over different time intervals.

In Cleveland, monuments are generally of marked stone or iron bars set below grade and protected by cast-iron boxes in paved street areas; or stone monuments or iron bars set at or near grade but not enclosed in boxes in areas other than pavements. This system rates high for permanence. Nevertheless, experience indicates measurable movement in even the best of protected street monuments. To determine the effect and possibly the amount of the movement of points, and the extent to which movement of points contributes to the difference of accuracies under study, the tabulation of accuracies of angles observed at different times was broken down according to the time interval between the two observations:

Time interval (yr)	Mean probable error (sec)
>0.5	7.3
0.5 to 2	6.4
2 to 3	9.2
3 to 4	6.5
4 to 5	6.6
5 to 10	5.3
10 to 15	7.6
Mean error for period	7.0

It seems clear, from the absence of any pattern in the foregoing and from the lack of consistent change with increasing time lapse, that either the movement over a long period becomes a compensating factor, or the error is smaller than the other contributing errors and, therefore, is lost among them. With Items (1) and (2) eliminated as determining factors in the variation of accuracies under discussion, only Items (3), (4), and (5) are left as determining factors.

The minimum difference between the two mean errors is 3.2″. Then three factors—Item (3) error of centering, Item (4) error of rod position, and Item (5) refraction—contribute to an increased mean error of 3.2″. It seems clear, then, that under the conditions of work stated herein there is a basic minimum error of angular observation of 3.2″.

It is to be noted that none of the three contributing factors to this basic minimum error are altered by (*a*) the calibration of the transit plate, or (*b*) the number of times that the angle is repeated at one setup. Therefore, when the basic error of 3.2″ is reached, neither precision of instrument graduation nor number of repetitions affects the actual result. Then, for city survey work that must be performed during all but the most extreme of weather conditions (assuming the use of sighting rods for pointings and plumb bob centerings of instruments), it would seem that the only advantage to a transit calibrated more closely than 1 min would lie in the fact that the limit of accuracy can be reached more quickly—that is, with fewer repetitions.

This leads to the question: How many repetitions are necessary to reach the ultimate accuracy fixed by external conditions, with a 1-min transit, that ultimate being, according to these tabulations, ±3.2″?

Previously herein it has been shown that the mean probable error of a single observation observed twice from the same setup (in each case an observation being the mean of six turns) is 3.8″. The mean probable error of the mean of the two observations from the same setup is then 2.7″, since the error of the mean is $r_m = \frac{r_s}{\sqrt{N}}$. It is to be emphasized that these are not presented as actual accuracies but merely the relationship of accuracies of angles observed by six turns as against angles observed by twelve turns, and all from the same setup.

It has also been shown that there is a 3.2″ minimum fixed error for the conditions and type of work studied herein, which error establishes the ultimate accuracy that can be obtained.

Since, as noted, the calculated probable error of six turns is 3.8″, and that of twelve turns is 2.7″, it should then require approximately nine turns to equal the limit of accuracy obtainable from one setup, with a 1-min transit, under the foregoing conditions and type of work.

In the work described herein, the average length of line and traverse side is 700 ft. The effect of the centering and rod position errors can be reduced by longer sights, but that is apart from the type or accuracy of transit used. An optical device such as is provided on some Swiss instruments, however, should materially reduce the centering error.

In a manner similar to that used in the investigation of the variation of the foregoing angular accuracies, the variation of linear accuracies is now

studied. As noted, the calculated accuracies of lines retaped at the same time are 37% greater than the accuracies of lines remeasured after an appreciable time interval.

The factors that might contribute to this variation are: Differences in tape lengths, movement of points, errors in tension on tape, and errors in determining temperature of tape. Again, except for possible movement of points, all factors tend to make the lower accuracy (lines remeasured with a time lapse) the more probably correct one.

On the accuracy of the tapes used, the writer has few authentic data. There should be a marked uniformity in tape lengths due to the type of tape used, but the manufacturer's statement of uniformity and accuracy has been the principal assurance. One dependable test was made, which showed an error of 0.0005 ft at 68° F, and 20-lb pull when suspended from the ends. Frequent tests by ordinary taping methods of new tapes against those which have been in service have never disclosed a variation of more than the thickness of a division mark, which is about 0.002 ft.

After these tapes have been in use for several months, however, they develop an exaggerated stretch for the standard pull when fully suspended, although they will continue to check for length when supported throughout at 10 lb. This is unquestionably due to the fact that the thickness of the tape has diminished through wear. When this error becomes noticeable, the tape is no longer used for anything but minor measuring.

To determine the extent to which the movement of points contributes to the variations under consideration, a breakdown of the measurement tabulation was made on the basis of the elapsed time, as in the study of angular accuracy variations. The results are as follows:

Time interval (yr)	Mean probable error, in parts per hundred
>0.5	0.00467
0.5 to 2	0.00405
2 to 5	0.00334
5 to 10	0.00389
10 to 15	0.00484
Mean error of period	0.00439

Again no pattern or variation is discernible in relation to time. It would seem, then, that the time element beyond an immediate remeasurement has no deciding effect upon the accuracy of remeasurements, and, therefore, the movement of monuments is either a smaller error than the other errors involved, or the movement is compensating over a period of time.

Errors of tension are now being corrected in a large measure by the use of a spring balance. Insufficient data are available to determine the amount of this error prior to the use of the spring balances.

Unquestionably the largest source of taping error, and probably the greatest contributor to the variations of accuracy under discussion, is the difficulty of determining tape temperature on sunny days. During the summer months pavements are often 15° warmer than the air only 4 ft above them. A tape that has been in the sun on a hot day will become too hot to hold. In winter

a tape will absorb heat from the direct rays of the sun in spite of low air temperatures.

No thermometer that the writer has used, which will stand daily use, can be depended upon for more than either air or pavement temperature. In general, when there is a wide discrepancy between the two, an attempt is made to estimate the temperature at tape level and tape temperature. Where a measured line passes through an area that varies widely as to sun and shade, some attempt is made to estimate the mean temperature for the line.

As previously indicated, the mean probable error of a single measurement of a pair of measurements made at the same time is 0.00319 part per hundred, and the mean probable error of a single measurement of a pair of measurements made at different times is 0.00439 part per hundred. Furthermore, it follows that the mean probable error of the mean of two measurements made after an appreciable time interval is 0.00310 part per hundred. From this it seems clear that a remeasurement at the same time will in actual fact not increase accuracy (omitting from consideration at this time an actual mistake of reading, marking, etc.), since the probable error of a single measurement is already within 3% of the accuracy of the mean of two measurements made at separate times.

Curve 6, Fig. 3(*b*), represents the accuracies of lines of all lengths that were repeated after an appreciable time lapse. Curve 7, Fig. 3(*b*), was derived from the same tabulation but was limited to lines of 1,000 ft and longer. In each case, the mean value of the probable errors included in the curves is indicated by a short horizontal line intersecting the curve at the level of the mean error. As determined previously, the mean probable error of curve 6 is one part in 22,800. The mean probable error of curve 7 is one part in 35,500. It is apparent from Fig. 3(*b*) that accuracies are approximately 50% greater for lines of 1,000 ft and longer than for lines of all lengths.

Examination of Fig. 4 indicates a marked similarity in linear and traverse accuracies in the normal ranges of precision. As noted, angular accuracies are about 30% higher. The similarity between the traverse accuracies and the lesser of the two contributing accuracies is in accordance with the probabilities. Although it is impossible to predict how linear and angular accuracies will combine within a single traverse (there are sixty-four possible combinations of the errors in a triangular traverse), nevertheless, in a large number of traverses the mean result can be determined.

To illustrate, the linear error of a traverse side and the angular error affecting that side can either be in such a direction that they are added, or in opposite directions and therefore subtracted. In a large number of traverses, the traverse errors are made up of large numbers of linear and angular errors, some of which compensate each other as the difference between the two, and some of which combine to the total of the two. The probability of one is equal to the probability of the other; therefore, the mean of the two extremes will represent the resulting traverse error for a large number of cases.

If the angular error is represented as E_a, the linear error as E_l, and the traverse error as E_t, then the traverse error (E_t) of a single traverse will lie between the two extremes, $E_l + E_a$, and $E_l - E_a$, assuming E_l is numerically greater.

The traverse error of a large number of traverses will be the mean of the two extremes, since the probabilities of each are equal:

$$E_t = \frac{(E_l + E_a) + (E_l - E_a)}{2} = E_l \dots\dots\dots\dots\dots (5)$$

—if E_l is the larger of the two contributing errors.

On this basis then, with the same angular accuracy as shown by curve 4, Fig. 4, and traverse sides of 1,000 ft or more as shown by curve 8, the traverse accuracy should increase, making the angular accuracy the determining factor of traverse accuracy.

As a check against this hypothesis, the traverses included in the tabulation for curve 8 were segregated as to the mean length of traverse sides. Twenty of the traverses in the tabulation had a mean length of side of 1,000 ft or more. The mean error of these traverses was found to be 0.00350 part per hundred or one part in 28,600. The mean value of curve 4 is one part in 29,400. The writer's hypothesis—that traverse accuracy is determined by, and equal to, the least accurate of the two contributing factors—is substantiated in this test by within 3%.

It would seem, then, that with the same equipment and methods now in use in the writer's office and described previously herein, a mean traverse closure of one part in about 30,000 can be maintained without undue consideration of climatic conditions, provided that traverse sides could be maintained at 1,000 ft and more.

It is of further interest to note that, for various accuracies, the percentage of the total number of traverses in rough terrain was as follows:

Traverse accuracies	Percentage of total in rough terrain
1 : 9,000 or less	100
1 : 14,000 or less	71
1 : 19,000 or less	55
1 : 20,000 or more	25

The traverses were studied for variations of accuracy based upon the over-all length of the traverse. Nothing conclusive was found, and the variations wavered back and forth about the curve for all lengths.

The dominating factor is length of side rather than length of traverse. The length of side in turn is determined by conditions of terrain. The tabulation does not contain a sufficient number of traverses of the same traverse side lengths to permit a study of the effect of variation of traverse over-all length.

Conclusions

This study has established certain findings of fact as to accuracies that have been obtained in actual survey practice. A number of these were itemized under the heading "Findings of Fact from Tabulations" (see Items (*a*) to (*i*)). Three other items follow from the discussion presented under the heading "Factors of Accuracy":

(*j*) The mean probable error of remeasurements of the same line with ordinary equipment is 0.00439 part per hundred, or expressed differently, the

difference between remeasurements of the same line averages 0.00920 part per hundred;

(*k*) The accuracy of measurement of lines of 1,000 ft and more is greater than that of lines shorter than 1,000 ft; and

(*l*) Accuracies of traverses with sides having a mean length of 1,000 ft or more are greater than those with sides having mean lengths under 1,000 ft, other factors being equal.

All the foregoing items support the following conclusions:

(1) There is a definite limit to the accuracy that can be obtained from a single setup of a transit. This limit is independent of the fineness of the transit's plate calibration, and of the number of repetitions in the observation; and it should be reached, by a good 1-min transit under average working conditions, at nine turns.

(2) Retaping of a line, forward and back, under average conditions, adds nothing to actual accuracy (except to locate mistakes in taping).

(3) In the long run, accuracies of traverses are determined by, and equal to, the lesser accuracy of the two contributing factors of taping and angular observation.

The foregoing findings (Items (*a*) to (*l*)) and reasoned conclusions ((1) to (3)) pose the following questions:

A. Present specifications for survey accuracy stress the error of closure primarily. In areas that lack a precise control, are such specifications adequate to assure a desired accuracy?

B. Is the trend toward transits of ever-smaller plate graduations justified for city work?

C. Might not accuracies of transits for city surveying be maintained at their present level, and work of setup and reading be facilitated by changes in design?

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
REPORTS

PHYSICAL PROPERTIES OF DRIVEN AND UNDRIVEN RIVETS OF HIGH-STRENGTH STRUCTURAL STEELS

PROGRESS REPORT OF THE COMMITTEE OF THE STRUCTURAL DIVISION ON STRUCTURAL ALLOYS

Synopsis

Results are given of behavior tests and strength tests made on various analyses of high-strength steel. Relationships are developed between shearing strength as used for the design of joints, and tensile strength as used in mill tests of material. Comparative data for standard carbon steel rivets are added.

Suggestions are given for the acceptance testing of any low-alloy steel proposed for use as rivets, and for the allowable unit stress in joints.

1. Development of Standard Specification

The Society has published a survey of "Structural Alloy and Heat-Treated Steels"[1] which indicated[2] the lack of a standardized high-strength rivet steel to be used in connection with silicon or other high-strength steels. (The more accurately descriptive term "silico-manganese steel" used in the Progress Report has made no headway against the commercially popular designation "silicon steel" for steels conforming to American Society for Testing Materials specifications (A. S. T. M. A94).) It also indicated that the American Society for Testing Materials was working toward the adoption of a specification for such a rivet steel.

As such a specification has passed through the tentative stage and has become an A. S. T. M. Standard, it is in order for the American Society of Civil Engineers to be advised of the developments in this field during the six years 1936–1942.

Note.—This Progress Report, entirely the work of Jonathan Jones, M. Am. Soc. C. E., member of the Committee, is published with Committee indorsement. Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **October 1, 1942**. Progress Reports are published in *Proceedings* only.

[1] Progress Report of Sub-Committee No. 2, Committee on Steel, of the Structural Division, *Proceedings*, Am. Soc. C. E., March, 1936, pp. 361–396.

[2] *Ibid.*, p. 394.

The A. S. T. M. committee had the privilege of studying the reports of some tests by the U. S. Navy on rivets made from four different heats of steel in the A94 category, and with a considerable range in hardness and strength. It was evident, from these tests, that increase in strength involved greater driving difficulties, and reduction in toughness or ductility. This study eventuated in both groups agreeing that the strength requirements should be placed just high enough to give a rivet-strength to plate-strength ratio for high-strength material, corresponding to that for ordinary structural carbon steel rivets and carbon steel plates (A. S. T. M. A141, A7, respectively).

In the production of rivet bars of these high-strength materials, it had been found that minimum and maximum tensile requirements on "as-rolled" bars resulted in unfair rejections. One bar from a given heat might be selected for a tensile test, and having been cooled in the middle of a pile (and therefore being well annealed) it might not reach the specified minimum tensile strength. Another bar, from a stronger heat, air-cooled in a draft, might harden to the point of exceeding the specified maximum strength. To eliminate this variable of cooling conditions (which, of course, meant nothing in a bar that was later to be re-heated and made into rivets) it was agreed to base the tensile requirements upon a fully annealed bar, thus establishing the fundamental properties of the steel. The specified tensile strength and yield point of the specification, now known as A. S. T. M. A195, may appear low until it is realized that they are based on fully annealed specimens.

TABLE 1.—DISTINGUISHING REQUIREMENTS OF "HIGH STRENGTH STRUCTURAL RIVET STEEL," A. S. T. M. A195

Description	Ladle analysis	Check analysis
(a) CHEMICAL COMPOSITION (PERCENTAGES)		
Carbon, maximum	0.30	0.35
Manganese, maximum	1.65	1.75
Phosphorus, maximum:		
Acid	0.06	0.075
Basic	0.04	0.05
Sulfur, maximum	0.05	0.063
Silicon, maximum	0.25	0.30
(b) TENSILE TESTS (TO BE MADE ON FURNACE-ANNEALED SPECIMENS)		
Tensile strength	68,000 to 82,000 lb per sq in.	
Yield point, minimum	0.5 tensile strength, and not less than 38,000 lb per sq in.	
Elongation in 8 in., minimum	$\frac{1,600,000}{\text{tensile strength}}$, and not less than 20%	

(c) *Upsetting Test.*—Replacing the more usual bend test, the upsetting test is performed on an unannealed specimen, and requires that specimens cut to a length of one and one-fourth times the diameter of the bar shall stand hammering down cold in a longitudinal direction to a length equal to three fourths of the original diameter, without showing seams or other defects that would tend to produce defects in the manufactured rivets.

After the new specification was published as tentative, it was found from tests, summarized in this report, that the tensile test requirements had been placed higher than was needed to insure the desired minimum rivet shearing strength, and high enough to run some risk of low ductility after driving. Therefore, the requirements were lowered slightly, and they now stand as in Table 1.

2. Confirmatory Riveting Tests

The fabrication and erection, in 1937, of two 350-ft two-span continuous plate girders for the Middletown-Portland (Conn.) Bridge created a need for the high-strength rivet in shop and field, and in this connection an opportunity was afforded to subject the new specification, then "tentative," to a set of "qualification tests" and other observations in shop and field. The results of these tests and observations, at the Pottstown, Pa., fabricating works of the Bethlehem Steel Company are summarized in the following. (More complete

TABLE 2.—Tests on Silico-Manganese Bar Specimens, Middletown-Portland (Conn.) Bridge
(Average Results)

Steel	Carbon C	Manganese Mn	Phosphorus P	Sulfur S	Silicon Si	Copper Cu	Yield point[a]	Tensile strength[a]	Elongation[b]	Reduction[c]
(a) Before Annealing (Mill Tests)										
Carbon, C	0.13	0.39	0.02	0.03	....	....	33,890	57,190	32.0	....
A195-H	0.26	1.44	0.04	0.04	0.10	0.26	57,800	86,970	23.0	49.1
A195-L	0.17	1.51	0.017	0.031	0.16	....	49,440	71,190	29.0	68.6
(b) After Annealing as Required by Specification A195–36T (Tests at Shop)										
A195-H	....	....	....	....	....	....	59,100	87,000	26.5	62.5
A195-L	....	....	....	....	....	....	43,950	70,400	31.3	70.6
(c) Original Specification Limits										
Maximum	....	....	....	....	....	....	....	85,000	20.0	....
Minimum	....	....	....	....	....	....	38,000	70,000	22.9	....

[a] Pounds per square inch. [b] Inches of elongation in 8 in. [c] Percentage reduction of section area.

details are contained in "Report on Qualification Tests of Silico-Manganese High Strength Structural Rivet Steel, Specification A195–36T" made to Subcommittee II of Committee A1 (Steel), American Society for Testing Materials, August 1938; of which a copy is on file with the Engineering Societies Library.[2a])

(*a*) In general, tests were scheduled to include duplicate specimens from the high (H) and the low (L) side of the specification range of tensile strength.

(*b*) Material was furnished in two bundles of $\frac{63}{64}$-in. diameter rivet stock.

[2a] 29 West 39th St., New York, N. Y.

Each bundle was from a different heat, one of which will be referred to as "high strength," or H-material, and the other of which will be referred to as "low strength," or L-material.

(*c*) Where for comparative purposes carbon steel bars or rivets were used, they conformed to the structural rivet steel specification, A. S. T. M. A141.

(*d*) The mill tests reported in Table 2 demonstrate that the L-bars were well on the low side of tensile strength, as desired, and the H-bars were slightly too strong.

Tests on rivets (as distinguished from the preceding tests on bars) were made as follows:

(*a*) Four straight rivets were made from each of the three steels. They were hot headed and air-cooled. These rivets were marked L_1, L_2, L_3, L_4, H_1, H_2, H_3, H_4, C_1, C_2, C_3, C_4. Rivets marked C were from the carbon steel, and those marked H and L were from their respective heats of silico-manganese steel.

(*b*) A steel driving block was made, comprising two duplicate halves, parting on the center line of four $1\frac{1}{16}$-in. holes. When bolted together, this block provided for the driving, and (after removing the bolts), for the removal intact, of four 1-in. rivets, with a 4-in. grip. For rivets H_1, H_2, L_1, L_2, C_1, and C_2, this driving block was warmed to approximately 125° F. For the remainder the block was chilled in ice water. The test rivets were heated in an oil-fired furnace to about 2,040° F (rivets C_1, C_2, C_3, and C_4, to 1,800° F),

TABLE 3.—Tensile Strength of Driven Rivets

Description	C_1[d]	C_2[d]	C_3[e]	C_4[e]	H_1[d]	H_2[d]	H_3[e]	H_4[e]	L_1[d]	L_2[d]	L_3[e]	L_4[e]
Tensile strength[a]	64.0	63.7	66.4	66.9	101.2	102.1	102.7	105.8	86.5	87.9	89.8	92.2
Average[a]	63.85		66.65		101.65		104.25		87.2		91.0	
Ratio to bar[b]	1.155		1.21		1.175		1.205		1.205		1.26	
Elongation[c]	40.5	43.5	40.5	40.5	9.3	18.8	6.2	9.3	37.5	37.5	31.0	31.0
Average[c]	42.0		40.5		14.0		7.75		37.5		31.0	
Ratio to bar[b]	0.755		0.73		0.33		0.185		0.735		0.605	
Reduction of area (%)	68.7	70.5	68.6	68.5	...[f]	36.7[g]	...[f]	...[f]	67.2	61.3	65.2	58.2
Average	69.6		68.5		36.7		....		64.2		61.7	
Ratio to bar[b]	1.08		1.06		0.61		....		0.90		0.865	

[a] Thousands of pounds per square inch. [b] Ratio to the corresponding value (not published) as found by check tests of the as-rolled bars, before heading them into rivets. [c] Inches elongation in two inches. [d] Driving block warmed to 125° F. [e] Driving block ice chilled. [f] Rivet broke at the swell of the neck, outside the gage marks. [g] Rivet broke 0.25 in. inside the gage marks.

and soaked at this temperature for one-half hour. They were then driven with a No. 90 riveting hammer and the rivets were bucked up with another hammer of the same size. These rivets were allowed to cool, and were then removed.

Specimens were washed in kerosene, wiped, dusted with talc, and examined for surface chilling defects, from which, however, all were free. Special grips were made for testing these rivets in tension, the pull being exerted under the shoulders formed by the two heads. Results of tensile tests on driven rivets are given in Table 3. The yield point was not pronounced and could not be

determined with any degree of accuracy. From the low ductility shown for the H-rivets it seemed that rivet steel on the "high" side of Specification A195–36T was under grave suspicion except for purposes where extreme loss of ductility is not important. Steel on the "low" side compared very favorably with carbon steel.

Filling of Holes.—Packs of 6-in. by $\frac{7}{8}$-in. plates were made, varying in length from 12 in. to 60 in., so that the packs varied in total thickness from $1\frac{3}{4}$ in. at each end through steps of $3\frac{1}{2}$ in. and $5\frac{1}{4}$ in. to 7 in. in the central 12 in. Plate material was silicon steel A. S. T. M. A-94. Two rows of holes were drilled from the solid, $1\frac{1}{16}$ in. diameter, on 3-in. pitch. Rivets and holes were numbered 1 to 20 on each pack and each row.

In one of the two packs, all twenty rivets in one row were C-rivets and all in the other row were H-rivets. In the other pack, all twenty rivets in one row were C-rivets and all in the other row were L-rivets.

Different driving techniques were used for the several rivets, these techniques being various combinations of variables as follows:

> Some rivets had straight and some had tapered shanks;
> Temperature ranged from 1,900° F to 1,975° F;
> Some rivets were driven "as heated";
> Some rivets had the points chilled after heating;
> Some rivets had both ends chilled after heating.

The principle involved in chilling points before driving is to induce the hotter and softer steel of the unchilled shank to upset and fill the hole before the formation of the new head can interfere with the flow of material into the hole.

After the rivets were driven, the pack containing one row each of C-rivets and L-rivets was planed down to the center of each of the two rows. The exposed rivet sections were then photographed. The same was done, with the "H" row only, in the second pack. The three photographs are reproduced in Fig. 1, each bearing an indication of the rivet steel used throughout the row photographed.

It will be seen that it is possible to fill the holes as perfectly with the silico-manganese rivets, of either grade, as with the carbon rivets. It should be admitted, however, that especial emphasis was laid on the filling of the 7-in. holes, and the packs shown in Fig. 1 are not the first ones made. Although the $5\frac{1}{4}$-in. and $3\frac{1}{2}$-in. lengths appear less satisfactory (in all three steels), this is simply because the same effort was not expended upon them as upon the longer rivets.

With the L-steel, chilling of points made a noticeable improvement in hole filling. With the H-steel, chilling only of points appeared to be better practice than chilling of both heads and points.

Joint Tests.—The purpose of these tests was to supply the data by which the writers of design specifications might prescribe allowable unit stresses on high-strength rivets in shear and bearing. Therefore only the L-heat, which has been shown to be at the low end of the allowable-strength range, was

employed, since design specifications should be based upon the lowest strengths likely to be developed by any steel conforming to the material specifications.

The details of the specimens, as well as the records of the test results, are shown in Table 4, corresponding coupon tests being reported in Table 5.

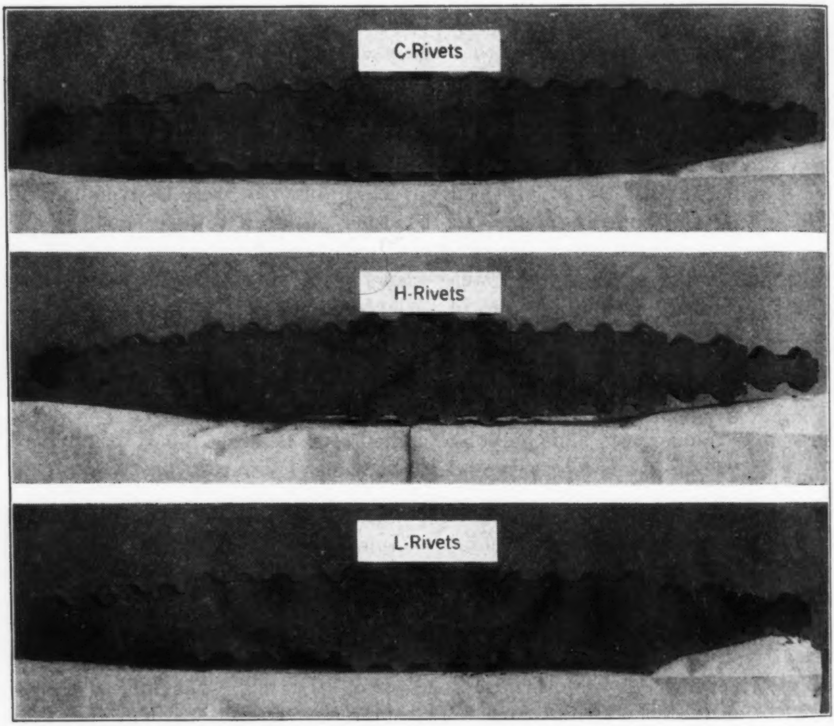


FIG. 1.—SECTIONS OF TESTED RIVET PACKS

Specimens LS1 are also called LB1, and were made and tested once and the results entered as part of each series. Before riveting, all contact surfaces were coated with graphite.

Shearing test specimens were made with one rivet, two rivets (in tandem), four rivets in tandem, and four rivets in two rows. This distinction in the case of the 4-rivet specimens was intended to develop whether the less equal distribution between rivets in tandem would affect the ultimate strength. It is evident that it did not, to any degree that need be taken into account in design.

The feature of these tests is the uniformity as between single-rivet and multiple-rivet specimens, the total spread in twelve specimens of four types being 5%.

Bearing test specimens varied the thickness and also, as the thickness decreased, increased the width and the end distance. Increasing the width eliminated net tensile area as a variable; increasing the end distance prevented

the rivet from shearing out the end of the plate. The objective was to find whether increase of bearing pressure, by itself, would affect the rivet shearing strength. The results show that with bearing pressure rising from 105,300 to double that value, the shearing strength of the rivet was indeterminately, if at all, affected.

It was evident from these tests that the writers of specifications might fairly regard 66,000 lb per sq in. as the minimum shearing strength of the driven rivets, in any usual design relationship to other factors. It also appeared that the bearing unit stress might be allowed to equal or exceed twice the rivet shear.

In none of the tests was there a pronounced drop of the beam to indicate a yield point. Stress-strain readings with two Ames dials, one on each side, were taken on all of the 4-rivet shear-test specimens, and on one in each of the remaining groups of 3. In Fig. 2 some of the load-deformation curves are plotted, superimposed, starting from 5 kips total load per rivet (1 kip = 1,000 lb), and plotting the average stretch of the two sides. At this value, with elongation equal to zero, the shear is 3.2 kips per sq in. The broken-line curve is a faired average of six tests on 4-rivet specimens LS3 and LS4, for which (in all cases) the ratio $\frac{s_b}{s_s} = 1.57$. The remaining curves are for single-rivet specimens, the ratio $\frac{s_b}{s_s}$ being as follows:

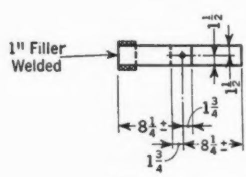
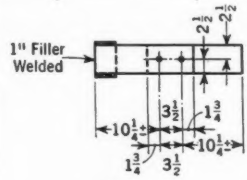
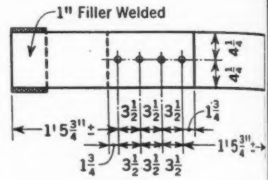
Specimen	Ratio
LB1	1.57
LB2	1.78
LB3	2.07
LB4	2.50
LB5	3.15

The individual curves are not very regular, as is natural considering the many factors entering into the progressive deformations. However, their correspondence with one another is satisfactory, and the unit shear, at which the rate of slope is twice the initial, appears to lie between 36 and 40 kips per sq in.

From these curves may be judged the fairness of the foregoing statement, that in practice the unit bearing may be allowed to equal twice the unit shear. There is inconsistency in the fact that LB5, with the highest bearing-shear ratio, deformed no more rapidly than LB1, with the lowest such ratio. The behavior of LB1 is confirmed by the plotting (shown herein as an average only) of six other 4-rivet specimens having the same bearing-shear ratio. Between LB3, however, with a bearing-shear ratio of 2.07, and LB4, with a ratio of 2.50, there is a marked decrease of resistance to flow. Somewhere in this region is apparently (despite the contradictory evidence of LB5) a reasonable value at which to limit the bearing-shear ratio in order that the breakdown of the joint under static loading may not be hastened.

After failure by shearing of the rivets, the elongation of the holes in the inner plates was determined by caliper reading to 0.01 in., greater accuracy than which would not have been justified. When compared with the ratio

TABLE 4.—Tests

		
*LS*1	*LS*2	*LS*3
Two Bars, 3"×5/8"×10"± Long	Two Bars, 5"×5/8"×1' 3 1/2"± Long	Two Plates, 8 1/2"×5/8"×2'6"± Long
One Bar, 3"×1"×10"± Long	One Bar, 5"×1"×1' 3 1/2"± Long	One Plate, 8 1/2"×1"×2'6"± Long

Specimen No.	*e* (in.)	*w* (in.)	*t* (in.)	Areas (Square Inches)			Ratio, $s_t : s_s : s_b$	Ratio, $s_b : s_s$
				Tension	Shear[c]	Bearing		
(*a*) Shearing								
LS1	1¾	3.00	⅝[b]	1.94	1.57	1.00	1 : 1.24 : 1.94	...
LS2	1¾	5.00	⅝[b]	3.94	3.14	2.00	1 : 1.25 : 1.97	...
LS3	1¾	8.50	⅝[b]	7.44	6.28	4.00	1 : 1.19 : 1.86	...
LS4	1¾	9.50	⅝[b]	7.38	6.28	4.00	1 : 1.17 : 1.84	...
(*b*) Bearing								
LB1[a]	1¾	3.00	1	1.94	1.57	1.00	1 : 1.24 : 1.94	1.56
LB2	2	3.28	⅞	1.94	1.57	0.875	1 : 1.24 : 2.21	1.78
LB3	2⅜	3.65	¾	1.94	1.57	0.75	1 : 1.24 : 2.57	2.07
LB4	2⅞	4.18	⅝	1.94	1.57	0.625	1 : 1.24 : 3.10	2.50
LB5	3½	4.94	½	1.94	1.57	0.50	1 : 1.24 : 3.88	3.15

[a] Specimen LS1 used for this test. [b] Two ⅝-in. bars and one 1-in. bar. [c] Based on a rivet diameter of 1 in.

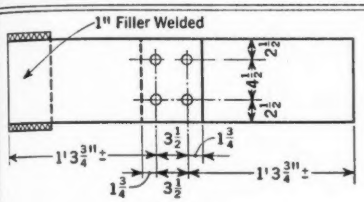
$\frac{\text{unit bearing}}{\text{unit rivet shear}}$, these elongations again indicated the propriety of the proposed 2 : 1 relationship.

In the case of the LS3 specimens (four rivets in tandem), there was less upsetting of the inner two than of the outer two holes (averages 0.045 in. and 0.095 in., respectively), recalling the unequal load distribution within the elastic range found by theory and experiment; but at the instant of failure all four rivets in these specimens parted simultaneously, indicating that the respective loads had equalized.

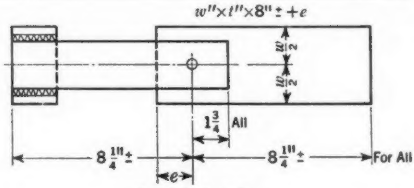
The suggested relationship, $\frac{\text{unit bearing}}{\text{unit shear}} = 2$, is quite low compared with the ratio of 2.67 authorized for carbon steel rivets and plates in the A. I. S. C. Specifications for Steel Buildings. The latter is proved to be proper by many experiments, including some made by the author and reported in a memorandum to the Committee of the American Institute of Steel Construction on Specifications.[2b] The reason probably lies in the fact that in the present

[2b] A copy will be supplied on request to the author, Jonathan Jones, Chf. Engr., Fabricated Steel Constr., Bethlehem Steel Co., Bethlehem, Pa.

GH-TENSILE RIVETS



*LS*4
Two Plates, 9"×5/8"×1' 9"± Long
One Plate, 9"×1"×1' 9"± Long



*LB*1, 2, 3, 4, 5
One Filler, 3½"×*t*×3" Long, Welded (*t*=Thickness)
Two Bars, 3"×5/8"×10"± Long, for All

TEST RESULTS, THREE SPECIMENS IN EACH TEST (LB PER SQ IN.)

Breaking load	Shear[c] Three specimens	Shear[c] Average	Bearing Three specimens	Bearing Average
CAPACITY				
104,000, 105,000, 107,000	66,400, 67,000, 68,050	67,150	....	......
212,000, 208,500, 206,000	67,500, 66,400, 65,600	66,500	....	......
423,000, 435,000, 415,000	67,400, 69,100, 66,000	67,500	....	......
430,000, 429,500, 428,000	68,300, 68,200, 68,050	68,200	....	......
CAPACITY				
104,000, 105,000, 107,000	66,400, 67,000, 68,050	67,150	104,000, 105,000, 107,000	105,300
108,000, 109,000, 105,000	68,800, 69,400, 67,000	68,400	123,500, 124,500, 120,000	122,700
105,000, 109,000, 106,500	67,000, 69,400, 67,800	68,100	140,000, 145,000, 142,000	142,300
101,000, 107,000, 105,500	64,400, 68,050, 67,300	66,600	162,000, 171,000, 169,000	167,000
110,000, 105,000, 100,500	70,700, 67,000, 64,100	67,300	222,000, 210,000, 201,000	211,000

high-tensile tests the contact surfaces of the plates were graphited. This graphiting was considered proper in determining the absolute minimum shearing strength of the rivets. It was probably too severe as to bearing, since the

TABLE 5.—COUPON TESTS OF MATERIALS

Stress, in lb per sq in.	Bar Thickness, in Inches: 1	⅞	¾	⅝	½	Annealed rivet bar
Yield point..........	50,100	....	52,200	42,900	....	43,950
Tensile..............	82,000	82,000	92,500	78,700	88,400	70,400

comparative absence of resistance to slip permitted deformation from bearing to commence earlier, and to become more pronounced. Since American specifications, for both buildings and bridges, require that shop contact surfaces be clean and not painted, and field contact surfaces while frequently painted provide much more friction than fresh graphite, it would seem that the allow-

able bearing pressure may in fact be higher than twice the shear; and its determination might better be based upon unpainted surfaces, or certainly surfaces on which the paint would be dry and not act as a lubricant.

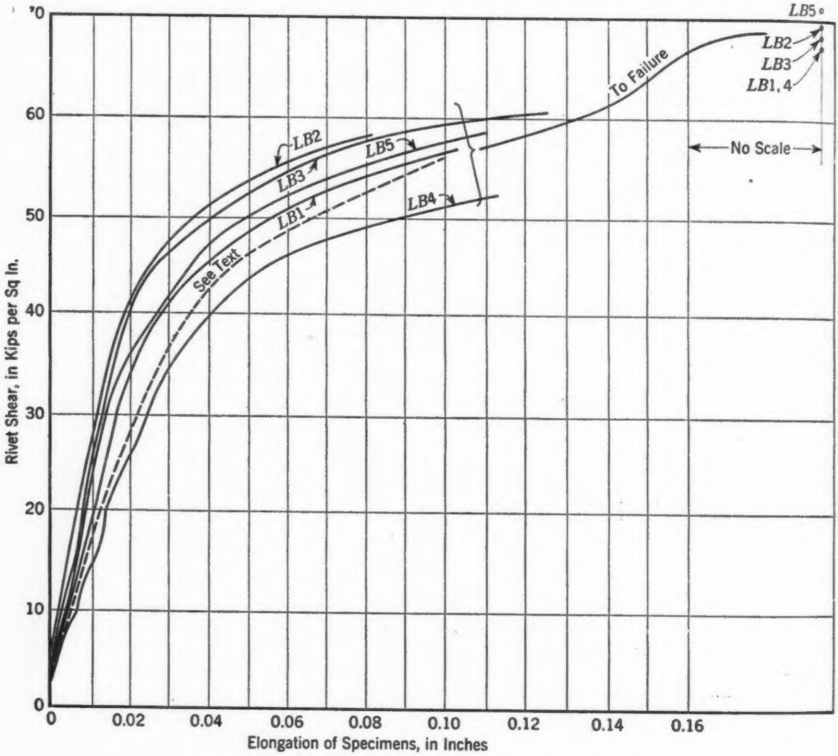


FIG. 2.—CURVES OF DEFORMATION OF RIVETED JOINTS

Age-Hardening.—After taking the photographs (see Fig. 1) of the planed-through hole-filling test packs, certain half rivets were removed and planed from the opposite side to produce a $\frac{1}{4}$-in. section through the center of each rivet. Each section was given a complete hardness exploration, using a Rockwell machine, for C-scale readings. Eleven months later the same sections were again hardness-checked, using points about $\frac{1}{16}$ in. away from the original points. This was to determine any age-hardening effect in the rivets, or progressive embrittlement. The small differences obtained at individual points in this test seemed very scattered. However, the averages (see Table 6) indicate that age-hardening for all

TABLE 6.—EFFECT OF AGE-HARDENING

No. of points	Type of rivet	Increased Hardness No.[a]	
		From:	To:
42	Carbon steel	5.4	6.8
43	High-strength L	13.0	13.65
42	High-strength H	26.1	26.1

[a] Average values

three steels is within the limits of error in determination, and negligible as an engineering consideration.

Impact Testing.—Impact tests were made about eleven months after driving, and the effect of age-hardening, if any, was therefore included in the impact results. Six rivets marked L11 and six rivets marked L12 were made and driven as follows:

All were headed by standard methods, normally cooled. The rivets marked L11 were heated to 1,450 F, soaked for 15 min, and then chilled in water. The rivets marked L12 were annealed by heating to 1,450° F, soaking for one-half hour, and cooling in the furnace overnight.

After heading, rivets were driven in the driving block described in presenting Table 3. Rivets were heated to about 2,050° F, and the driving block chilled in ice water. Then the rivets were driven by a hand riveter. Only two rivets were heated and driven at a time, as these two heated the block to such an extent that additional rivets would have been driven in a warm block instead of a chilled one. After each pair of rivets was driven, they were allowed to cool in the block 15 min and were then removed. The block was rechilled for the next driving.

Six rivets H11 and six rivets H12 were made and driven, duplicating the foregoing procedure for L-rivets.

An "alternate-standard" three-notch round Izod specimen (0.450-in. diameter) was machined from each of the rivets from the H11, L11, H12, and L12 sets. In this type of specimen the first notch is $1\frac{1}{8}$ in. from one end, the second notch $2\frac{3}{16}$ in. from the end, and the third $3\frac{1}{4}$ in. from the end. By keeping the first notch identified as nearest the driven end, it was possible to record the impact strength of the various specimens at a distance of $1\frac{1}{8}$ in.,

TABLE 7.—IMPACT TESTS IN ELEVEN MONTHS

Rivets[a] (1)	First notch (2)	Second notch (3)	Third notch (4)	Rivets[a] (1)	First notch (2)	Second notch (3)	Third notch (4)
(a) RIVETS HEATED TO 1,450° F (15 MIN) AND CHILLED IN WATER, PRIOR TO USE							
H11	17	22	35	L11	52	43	89
(b) RIVETS HEATED TO 1,450° F (30 MIN) AND FURNACE COOLED, PRIOR TO USE							
H12	14	20	25	L12	42	65	74

[a] Average of six observations each.

$2\frac{3}{16}$ in., and $3\frac{1}{4}$ in., approximately, from the driven end of the rivets from each set. The average results (see Table 7) are summarized in the following paragraph.

In both the "11" and the "12" sets the L-rivets gave higher impact values than the H-rivets. Both L-rivets and H-rivets showed higher impact strength as the distance of the notch from the driven end increased.

There was not much difference in the impact strength of the "11" and "12" groups. This is in line with what might have been expected, because

the heating to 2,050° F for riveting would naturally minimize the effect of the different pretreatments. Accordingly, it seems unnecessary to specify annealing of hot-headed rivets after manufacture. The impact strengths for the L-rivets were all entirely satisfactory, and those for the H-rivets might be questioned for some applications.

Fabrication Experiments.—This program was carried out, at the Pottstown Works, in fabricating the two center girders of the west approach, Middletown-Portland Bridge, using H-rivets in girder G3A-A and L-rivets in girder G3A-B, all rivets being 1 in. in diameter, tapered. The two girders were practically alike, 86 ft long, $8\frac{1}{2}$ ft deep, having field splices at the ends to connect to two 130-ft girders, and being the center sections of 346-ft spans continuous over three supports. The main material was silicon steel. Variables were tested on the web rivets near the ends, where there were web plates, side plates, and chord angles, driving through three and five thicknesses of material.

All rivets in these girders were heated in oil-fired furnaces, driven with an 80-ton "horseshoe" riveter. General practice for 1-in. carbon rivets, driven with this machine, showed a rivet temperature over a wide range, between 1,150° F and 1,500° F, at the instant of driving. This temperature variation was due to several conditions. A rather hot furnace was used and, to reduce delays while waiting for rivets to heat, some rivets were taken out of the furnace at the lower temperature. Rivets that stayed in the longest came out at the higher temperature. When delays were encountered either in inserting the rivets or in getting on them with the machine, the hotter rivets often had a chance to cool down to the lower temperatures. Another standard practice is to have a few rivets stuck ahead of the riveter, these rivets being removed from the furnace and placed on the girder with a shovel; again, the last rivets to be driven will have cooled off considerably. The lower of the foregoing range of temperatures (1,150° F) is an approximation, being below the range of the optical pyrometer used. The wide temperature range, with the heavy riveters, produced good results, both from the standpoint of satisfactory rivets and of costs.

Driving of the high-strength rivets required that rivet temperature be held more uniform. The lower limit was determined by the capacity of the riveter to upset the rivet complete in one stroke. For the H-rivets, the temperature was about 1,350° F, and slightly lower for the L-rivets. If machine rivets are driven too hot, the riveter must stay on the rivet longer to allow it to cool, in order to leave tight rivets. These two conditions practically determined the heating range at between 1,350° F and 1,500° F for the H-rivets.

Due to the harder material in the special rivets, the actual time required to upset a rivet was increased, quite noticeably with the H-rivets. In addition, some of the longer rivets required two strokes of the riveter, the screw being tightened for the second stroke. This also was due to the harder rivet material.

The driving conditions were controlled so that the number of cutouts was no greater than for carbon rivets in the other girders. This was obtained at a sacrifice in rivets per day, or in costs per hundred. The following comparison of cost is made:

Kind of rivets	Rivets driven (8 hr	% cost
Carbon	1,950	100
"L"	1,550	126
"H"	1,430	137

Furthermore, certain variables were introduced into the conditions for driving the H-rivets and L-rivets in certain locations on the girders, to determine their effect both technically and as to cost, as follows:

For these six variables, groups of about thirty rivets each were selected on each girder in the web toward the end, some through the web plate and two side plates, and others through the web, side plates, and chord angles. One variable condition was tried in each group. The results are shown in Table 8. Rivets were tapped for looseness as soon as they were cold. No additional loose rivets were found the next day.

In the foregoing special tests (Table 8), loose rivets are reported per unit of about thirty rivets. Cutouts throughout the girders observed averaged less than one per hundred rivets. General practice, therefore, produced results equal to the best results from the special techniques.

These tests emphasized the necessity of using well-heated rivets. Annealing the rivets showed no particular advantage. Chilling the points of rivets having a maximum grip of $4\frac{1}{8}$ in. was not advantageous. Some of the very long rivets through the stiffeners in the center of the girder were chilled, to advantage, so the foregoing test on short grip rivets should not condemn the practice of chilling the points of machine-driven rivets under extreme conditions.

Proper bolting up of material is necessary, but the amount varies with different types of work, straightness of material, etc. The foregoing test with all holes bolted (see Table 8), of course, gave assurance against loose rivets,

TABLE 8.—Effect of Variables in Driving Conditions

Group	Driving Conditions	Loose Rivets	
		H	L
1	Annealed at 1,450° F and furnace cooled. Rivets were then reheated to driving temperature and driven	1	1
2	Soaked one hour at 1,450° F and driven	0	0
3	Heated to about 1,250° F and driven. Rivets hit twice by the machine	3	2
4	Heated to about 1,450° F and the point of each rivet dipped 1 in. in water for 5 sec before driving	2	2
5	Bolt placed in every hole and drawn up tight. Rivets were then driven replacing these bolts one at a time	0	0
6	Each rivet hit by the machine three times and after the last stroke the pressure was held on the rivet 10 sec	2	2

but such bolting is not necessary on any usual run of work. Excessive driving did not appear to show any advantage over normal driving.

In summarizing, attention is called to the difference in the results between the H-rivets and the L-rivets, both under the same specification, as evidenced not only in costs of driving, but in ductility and hardening. General statements regarding comparisons of carbon steel and L-rivets do not hold when considering the H-rivets. The latter were considered by the engineer who supervised the shop work to be too hard to be satisfactory.

A bar of H-material heated to 1,450° F and water quenched will, under bend test, break without any appreciable bending. A water-quenched specimen of L-material stood about 45° of bend, but even with this material the break indicated a hard structure. No attempt was made to shear quenched specimens, because of this hardness. As shearing of rivets is often necessary, this hardening property due to possible accidental quenching, particularly as to the H-material, is a disadvantage.

The hardness of the H-rivets after driving was also noticed when a few rivets had to be cut out. Removal of heads by either the rivet buster or a "B and O" punch and maul resulted in breaking off the heads without warning, with resulting danger to men working in the vicinity.

As previously noted, both L-rivets and H-rivets cost more to drive than carbon rivets. Theoretically there should be only 75% as many high-strength rivets required; but most of the rivets in girder work, for example, are spaced by specifications which set a maximum spacing, and comparatively few are spaced only for strength. It is reasonable, therefore, to estimate that the number of high-strength rivets required would be about 90% of the carbon rivets, and since they cost about 31% more per 100 to drive, the net riveting cost might be 18% greater for high-strength rivets.

Punching, reaming, and drilling would be reduced between 5% and 10% by using the reduced number of high-strength rivets. Considering this fact, the total cost of riveting, punching, reaming, and drilling would still be about 3% cheaper if carbon rivets were used.

Therefore, as far as shop fabrication is concerned, economy in the use of the high-strength rivet seems to be limited to those cases (such as bridge chord splices and the like) in which the extra strength ($\frac{1}{3}$ to $\frac{2}{3}$) of the rivet can be utilized to the fullest extent in design, and will reduce the total number of rivets in the group investigated by at least 20% to 25%.

All these high-strength rivets in the test girders were machine driven, except the comparatively few driven by hand in the special tests program. Therefore, comparative costs of hand driving cannot be derived from the data reported.

Field Driving Tests.—The 350-ft continuous girders in the Middletown-Portland Bridge required field splicing, and it was arranged that one girder should be driven with rivets from the L-heat, and one girder with rivets from the H-heat of steel used in the foregoing tests. Observations were made on the driving time and other behavior of these rivets, as reported in Table 9.

TABLE 9.—FIELD DRIVING TIME STUDIES (H = HIGH STRENGTH; L = LOW STRENGTH)

Rivet length[a] (in.)	First Gang				Second Gang			
	Type H		Type L		Type H		Type L	
	Heat[b]	Dr[c]	Heat[b]	Dr[c]	Heat[b]	Dr[c]	Heat[b]	Dr[c]
3¾	170	41	155	24	180	21	155	38
5¼	180	29	170	28	180	23	170	22
6¼	180	31	170	24	180	26	150	24
7⅜	180	34	156	24	180	27	155	23
8⅝	190	36	150	20	200	27	150	25
9⅝	180	36	175	23	240	25	180	29

[a] All lengths greater than 3¾ in. had tapered necks under the "manufactured" heads. [b] Number of seconds required to bring a single rivet to driving heat in the coal forge. [c] Average time for driving one rivet (usually averaged over about twelve rivets).

There were no cutouts, and the average count per 8-hr day was 297 rivets. Fitting up was very thorough. Points were dipped one second to chill them before driving. Indications were that any good gang could be trained to drive the rivets, but the H-rivets required hard pushing and holding and caused some pain or even swelling in the hand.

Later, similar observations were made on reasonably comparable carbon steel rivets. The results indicated that with other factors (grip, accessibility, number, experience, etc.) equal, the silico-manganese rivets should cost about 20% more in the field than the usual carbon steel structural rivet.

Economics.—A rough comparison of the economics of the use of this rivet in bridgework may be obtained as indicated in Table 10. Although these

TABLE 10.—RELATIVE COSTS (DOLLARS) OF USING STRUCTURAL STEEL AND SILICO-MANGANESE STEEL RIVETS

No.	Material	⅞-In. Rivets		1-In. Rivets		1⅛-In. Rivets	
		Shop	Field	Shop	Field	Shop	Field
	(a) Standard Carbon Steel Structural Rivets						
1	1-lb at 3¢	0.03	0.03	....	....	....	....
2	1.5-lb at 3¢	....	....	0.045	0.045	....	....
3	2-lb at 3¢	....	....	....	....	0.06	0.06
4	Cost of hole and rivet driving	0.055	0.235	0.06	0.255	0.065	0.28
5	Total, Items 1 to 4	0.085	0.265	0.105	0.30	0.125	0.34
	(b) Additional Cost for High-Strength Silico-Manganese Steel Rivets						
6	1-lb at 1¢	0.01	0.01	....	....	....	....
7	1.5-lb at 1¢	....	....	0.015	0.015	....	....
8	2-lb at 1¢	....	....	....	....	0.02	0.02
9	Add 30% for shop rivets	0.015	....	0.018	....	0.019	....
10	Add 20% for field rivets	....	0.047	....	0.051	....	0.056
11	Total, Items 5 to 10	0.11	0.32	0.14	0.366	0.164	0.416
12	Ratio, Item 11 : Item 5	1.29	1.21	1.33	1.22	1.32	1.22

values are only approximations, they indicate that unless one fourth or more of the rivets can be eliminated from a given splice or joint, it will not be economical to substitute silico-manganese for carbon rivets. These ratios of rivet elimination can be attained, and therefore the rivet has a place, in bridgework; but only in exceptional cases, and subject to a more exact analysis of each case proposed.

Such an "exceptional" case arose in the erection of the bridge over the Mississippi River at Natchez, Miss. Because of the cantilevering procedure adopted for erection, certain joints become over-stressed in rivet shear, for the erection condition only. By changing the rivets in such joints from carbon (A141) to high strength (A195), the same number of rivets sufficed and there was obviated any increasing of the length or width of splice and gusset plates.

There have been, and will be, many applications of the high-strength rivet in the flanges of heavy silicon steel girders, regardless of the apparent lack of

economy in so doing, because the unknown bending stresses on long-grip rivets may justify the precaution of employing rivets of enhanced strength.

Outcome of Test Program.—It could be concluded fairly from the foregoing data that:

(1) Rivets made from steel on the low side (L-rivets) and the high side (H-rivets) of Specification A195-36T, successfully passed all tests, except that H-rivets were—

(*a*) Deficient in ductility after driving,
(*b*) Questionable in impact value after driving,
(*c*) Expensive to drive, and
(*d*) Hard on the workmen.

(2) The inherent shearing strength of the L-rivets after driving was not less than 66,000 lb per sq in.

Therefore the tensile strength (annealed) in the then "tentative" Specification A195 was reduced by action of A. S. T. M. from 70,000–85,000 to 68,000–82,000 (as listed in the discussion introducing Table 1), in the belief that such a steel would be equally entitled to working stresses appropriate to "silicon" (A94) and other structural steels of 45,000–50,000 minimum yield point, and would at the same time be more economical and safe of application.

3. Low-Alloy Compositions

During the period through which the silico-manganese rivet was being tested and its specification established, favorable reports were publicized of rivets made of some proprietary steels, about the same in strength as silico-manganese, but more expensively alloyed. Also, opinions were advanced that certain alloying elements which might be employed to enhance the strength, might be adverse to good riveting practice—in fact, might make tight riveting impracticable.

The author decided, accordingly, that there were two reasons why Specification A195 should be broadened to provide for various low-alloy types of steel:

> First, if another type of rivet steel could be developed, which would have the strength of the silico-manganese type herein tested, and much greater ease in driving (say only 5% to 10% more costly to drive than carbon steel), such a rivet would have an important economic application, and its cost per pound would not enter materially into the question of its economy.
>
> Second, it seemed probable that there were steels available which would pass the physical tests of Specification A195, but which could not be used for rivets. Therefore, it was the author's opinion that another series of tests, called perhaps "Qualification Tests," would have to be imposed upon any particular composition offered, in order to prove its riveting characteristics.

Such a program of "Qualification Tests" was established, and during 1939 the Bethlehem Steel Company procured steels of a variety of compositions and conducted the tests. Contrary to what was anticipated as at least a possibility, this research did not reveal any conclusive reasons for the rejection of

any of the types examined, nor any clear rating in order of superiority. Neither did it demonstrate that any of the low-alloy types tested should be preferred to the silico-manganese rivet steel A195 from a standpoint of easier or more foolproof heating and driving.

In Table 11, the compositions and physical properties of the steels in this research are stated. Item 1, Table 11, is the ordinary carbon rivet (A141),

TABLE 11.—COMPOSITION AND PHYSICAL PROPERTIES, BAR MATERIAL, ANNEALED AT 1,450° F

No.	Composition (Check Analysis)										Physical Properties[a]			
	C	Mn	P	S	Si	Ni	Cr	V	Mo	Cu	Lb per Sq In.		Elongation[b] (in.)	Reduction (%)
											Yield point	Tensile strength		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	0.16	0.44	0.01	0.03	0.01	0.02	0.02	0.02	0.00	0.24	33,700	53,050	36.0	66.8
2[c]	0.17	1.51	0.02	0.03	0.16	...	...	...	...	...	43,950	70,400	31.3	70.6
3[c]	0.26	1.44	0.04	0.04	0.10	...	...	...	...	0.26	59,100	87,000[e]	26.5	62.5
4	0.09	0.67	0.10	0.03	0.12	0.28	0.25	0.02	0.00	0.55	44,400	63,400[f]	35.0	71.8
5	0.09	0.75	0.09	0.03	0.35	0.45	0.64	0.02	0.00	0.61	50,300	72,850	34.4	72.9
6	0.11	0.34	0.13	0.03	0.59	0.10	0.87	0.00	0.00	0.42	50,800	70,450	34.2	69.8
7	0.14	0.37	0.05	0.03	0.14	1.96	0.00	0.00	0.01	0.96	56,300	71,400	31.0	64.4
8	0.11	0.72	0.01	0.02	0.03	1.30	0.00	0.00	0.12	1.57	52,000	69,650	30.0	63.7
9	0.11	0.74	0.11	0.03	0.06	0.75	0.00	0.00	0.01	1.28	59,400	74,000	30.1	65.9
10	0.27	1.60	0.02	0.03	0.19	0.02	0.02	0.02	0.00	0.03	49,900	83,500[e]	28.1	64.0
11	0.14	1.18	0.03	0.03	0.65	0.02	0.46	0.00	0.00	0.01	47,050	71,250	33.5	73.5
12[d]	0.17	1.25	0.03	0.04	0.11	0.06	0.05	0.12	0.00	0.06	.....	.....	...	...
13	0.18	1.35	0.03	0.04	0.26	0.13	0.14	0.10	0.03	0.21	54,700	79,300	29.8	69.3
14	0.15	0.50	0.03	0.02	0.16	1.88	0.05	0.03	0.19	0.15	50,240	69,800	30.2	66.3

[a] Usually an average of five tests, although sometimes four tests. All specimens passed the 180° bend test except Items 2 and 12, for which the test was not taken. [b] Inches elongation in 8 in. [c] These are the L and H silico-manganese heats tested in 1937 and reported previously in this paper. [d] Ladle test reports only. [e] Exceeds upper limit of A. S. T. M. Specification A195. [f] Under lower limit of A. S. T. M. Specification A195.

included as a basis of reference. Items 2 and 3 are the "low" and "high" heats, respectively, of silico-manganese steel, which had been tested previously and reported hereinbefore. Additional tests for this type of steel were not made in the second program.

Steel 10, Table 11, is not a "low-alloy," but a still harder grade of silico-manganese than Item 3. It is too hard to be acceptable under Specification A195, but was included for further information on the silico-manganese type. Steels 4 to 14, except Item 10 (ten types in all), were obtained in the market. Most of them will be recognized on inspection as being commercially available proprietary compositions, some of which have had extended use; it will be seen that the heats used in these tests are at the soft end of the advertised compositions and therefore appropriate for rivets.

In connection with Steel 14, Table 11, it may be well to state that every effort was made to locate a heat of true SAE 2115 (1.50% nickel) conforming to the minimum tensile requirements of Specification A195; this proved to be

unobtainable at the time, and it was necessary therefore to use a heat with, as will be noticed, a "moly" content which can scarcely be classed as negligible.

Transformation Temperatures.—Table 12 shows, for each of the steels examined, certain heating and cooling phenomena which it was felt might be important, but which have not definitely proved to be so. There has been speculation as to whether a low cooling-transformation range, particularly in connection with a high degree of expansion or recalescence, would tend toward poor gripping action, inasmuch as hammering would be stopped before the rivet had cooled below this point of temporary expansion.

TABLE 12.—HEATING AND COOLING PHENOMENA
(Rate, 400° F per Hr)

No.	Transformation Temperatures				Contraction in mils per in.[d]
	Heating		Cooling		
	Beginning	End	Beginning	End	
	(15)[c]	(16)	(17)	(18)	(19)
1	1,370	1,560	1,480	1,300	7.5
2, 3[a]	1,360	1,550	1,360	1,200	5.0
4	1,360	1,640	1,500	1,050	5.4
5	1,380	1,650	1,485	1,250	6.6
6	1,450	1,700	1,600	1,315	6.2
7	1,300	1,490	1,340	1,180	6.7
8	1,300	1,500	1,320	1,200	6.2
9	1,390	1,620	1,420	1,250	5.7
10	1,330	1,500	1,330	1,180	7.8[e]
11	1,400	1,630	1,470	1,240	6.1
12[b]	....	....	....	....	...
13	1,345	1,550	1,375	1,160	7.7[e]
14	1,320	1,510	1,340	1,200	7.7[e]

[a] Similar steel, but of a different heat. [b] Test not made. [c] Column numbering continuous with Table 11. [d] Cooling from 1,700° F to 800° F. [e] Somewhat doubtful because of extrapolation.

For carbon rivets this range is given as 1,480° F to 1,300° F (Cols. 17 and 18), and much lower (1,360° F to 1,200° F) for silico-manganese. Could it be because of this fact that the latter steel is required to be driven some 30% longer than the former in order to be equally tight and to stay tight?

Table 12 shows that this range is below that for carbon steel, in the case of most of the low-alloy steels, but is higher than that for carbon in the case of a few. This information, however, is not directly applicable. It was obtained in the laboratory for heating and cooling rates of 400° per hr. For the rapid, and variable, heating and cooling rates of riveting practice, these transformation points could be very different from those in Table 12.

Naturally, when driving is terminated, the rivet temperature depends upon the rate of extraction of heat from the rivet by surrounding material, and can be quite variable. However, as an approximation, the contraction has been measured on the dilatometric curves, from 1,700° F to 800° F. It will be seen from Col. 19, Table 12, that the differences are insufficient to affect, noticeably the clamping power of the rivets.

However, this question of differences in loosening being due to the transformation point will scarcely be answered unless and until some additional tests are made, in which the temperature from which cooling starts will be as high as that of heated rivets, and in which the cooling rate will be faster.

Hardenability and Drive Hardening.—The tests summarized in Table 13 were made (*a*) to determine the hardening effect of driving rivets into the work

and (*b*) to correlate this effect if possible with the hardening of bar stock in an air blast. The latter could then be introduced into the specifications as a mill acceptance test of the material, if the correlation tests should show that it would give satisfactory, and necessary, control on hardening of actual rivets.

Results of the air-blast quench from 1,850° F on rivet bars are given in the form of ratios of properties after, to the same properties before, the quench. Col. 20, Table 13, expressing this ratio for Rockwell B hardness, and Col. 21, expressing it for tensile strength, are similar, as might be expected. Col. 22 shows reduction of elongation by the quench.

One-inch rivets were driven in groups of four, into a split-and-bolted block affording a 4-in. grip, the block being hand warm. After removal from the block, these rivets were pulled in special grips which engaged the under side of the heads.

The ratio expressing loss of elongation of the driven rivet as compared with the annealed bar (Col. 23, Table 13) is strikingly similar to Col. 22. The

TABLE 13.—HARDENABILITY AND DRIVE HARDENING

No.	HARDENABILITY (BAR QUENCH TEST) Ratio, Air Blast Quenched from 1,850° F / Annealed			DRIVE HARDENING (NO CHILLING) Ratio, Driven Rivet / Annealed Bar			Shear strength of driven rivet; ratio[b] $\frac{s_s}{s_t}$
	Rockwell B hardness	Tensile strength	Elongation in 8 in.	Elongation in 2 in.	Tensile strength	Actual elongation in 2 in. (%)	
	(20)[a]	(21)	(22)	(23)	(24)	(25)	(26)
1	1.20	1.18	0.75	0.76	1.16	42.0	....[c]
2	...[c]	...[c]	...[c]	0.78	1.21	37.5	0.96
3	...[c]	...[c]	...[c]	0.33	1.18	14.0	....[c]
4	1.14	1.16	0.67	0.66	1.26	36.5	1.025
5	1.17	1.29	0.57	0.59	1.39	31.8	1.065
6	1.12	1.17	0.59	0.58	1.26	31.5	1.055
7	1.02	1.05	0.79	0.73	1.11	37.3	0.90
8	1.15	1.34	0.70	0.83	1.12	34.7	1.025
9	1.04	1.08	0.76	0.84	1.06	40.8	0.93
10	1.13	1.25	0.51	0.52	1.23	25.0	0.90
11	1.15	1.29	0.52	0.53	1.34	29.8	0.995
12[c]	...	...	...	...	...	...	....
13	1.13	1.28	0.63	0.57	1.31	28.0	1.025
14	1.16	1.25	0.59	0.70	1.29	33.8	1.02

[a] Column numbering continuous with Table 12. [b] s_s = shear strength of driven rivet and s_t = tensile strength of annealed bar. [c] Test not made.

increase of tensile strength (Col. 24) is similar to Col. 21. Therefore, the air-blast quench would be a fair criterion of what to expect in the way of drive-hardening, if such should be required.

Tests were also made of driven rivets in shear. For each steel, three duplicate specimens were made of each of four patterns: One rivet, two rivets in line, four rivets in line, four rivets in a square. All were double shear specimens made of high tensile plate, designed to fail in the rivets, and with contact surfaces graphited to minimize friction as an influence.

The assumed objective was a specification which would insure a minimum shear strength of driven rivet of 64,000 lb per sq in., based on the cold rivet diameter; but of course, such a requirement is no part of a practical mill acceptance test for the bars. Since the minimum annealed tensile strength in Specification A195 as now written is 68,000, it is desirable that the ratio—

$$\frac{\text{shear strength of driven rivet}}{\text{tensile strength of annealed bar}} \text{ shall exceed } \frac{64}{68} \text{ or } 0.94.$$

Examining Col. 26, Table 13, it is found that for all but three of the steels tested this ratio proved to exceed 0.94. For the three types in which the ratio is less, a user might well require that the minimum tensile strength be kept above 68,000, in order to be certain of obtaining 64,000 in rivet shear.

These comparative results indicate that if and when, under Specification A195, some composition other than silico-manganese is offered, the air-blast quench test offers a cheap and adequate means of determining how its strength and elongation will be affected by use as a rivet; but that only a direct test of joints in shear will determine its shearing strength, as a rivet.

It is desirable, of course, that the elongation after driving shall not be too far reduced. Table 13 shows (Col. 23) that Steel 11, with only 71,250 annealed tensile strength, retains only 53% of its annealed bar elongation, after driving. Steels 5 and 6, with annealed tensile strengths of 72,850 and 70,450 respectively (that is, rather soft) retain 59% and 58%, respectively, after driving. Steel 13, the annealed tensile strength of which is 79,300, or near the upper limit permissible, retains more (63%) of its annealed bar elongation than the softer steels just mentioned. As between different types of steel, therefore, the tensile strength is not an indication of the relative loss of ductility after driving.

The least actual elongation in 2 in., after driving, for any steel among the low alloys considered was 28.0 (Steel 13); the greatest was 40.8 (Steel 9). The higher strength silico-manganese (Steel 3) showed in the earlier work so little ductility in the driven rivet (14.0%) that it led to lowering the permissible tensile in Specification A195 from 85,000 to 82,000. Steel 3 is thus not an acceptable steel under the present specification; neither is Steel 10. The steel of same type at the soft end of the specification (Steel 2; annealed tensile strength, 70,400) showed excellent ductility after driving (37.5%).

Although there is here a great variation in the ductility after driving, and some engineers will feel a preference for the steels that show the higher values, it is doubtful whether this feature should be a determining factor. Heat-treated alloy steels showing 20% elongation or less, in 2 in., are giving satisfactory service under all manner of shock loads. As none of the steels tested, which comply with Specification A195, showed less than 28% in 2 in. for driven rivets, it would seem that all are satisfactory riveting steels in this respect.

Impact and Age-Hardening.—The information recorded in Table 14, Cols. 27 and 28, is for the intermediate notch of a three-notch, round Izod specimen, and is typical of other information obtained. Each value is the average of six breaks. Duplicate specimens were tested after six months to reveal any effect of aging, but there was nothing beyond the range of experimental errors.

The test is not seen to have significance as to the service behavior of structural rivets. This may be regarded as statistical information, presented because it is available, but not indicating a necessity for further research.

Scaling and Hole Filling.—This was intended to be a practical test of shop behavior. For each steel a pack of plates was made up, starting with two plates 6 in. by $\frac{7}{8}$ in. by 5 ft, then adding shorter plates of the same thickness in symmetrical pairs, until at the center of the pack there were eight thicknesses in all, or a 7-in. grip, for nominal 1-in. rivets. In all, the pack was drilled for forty rivets, or two lines of twenty each.

The rivets through eight plates, and half of the rivets through six plates, had tapered shanks such as are frequently used in bridge building; the others were cylindrical. All holes were cylindrical.

These plate packs were thoroughly bolted for riveting, and then successive packs were driven with No. 90 hammers. The same riveting gang was used throughout, and the gang had no knowledge of what steel was being driven except in the case of the plain carbon steel which, of course, revealed itself by its relative ease of driving. Driving temperatures and length of time of driving were kept close to that which seemed to be optimum for each steel, and were recorded.

TABLE 14.—IMPACT, AGE-HARDENING, SCALING, AND HOLE FILLING

No.	Izod, Ft-Lb: Soon after driving (27)[a]	Izod, Ft-Lb: 6 months after driving (28)	Weight of scale from 8 rivets (carbon =1.00) (29)	Rating for hole filling[b] (30)
1	94	91	1.00	1
2	..[c]	52	...[c]	2
3	..[c]	17	...[c]	4
4	44	35	1.10	2
5	29	28	1.30	4
6	17	21	2.80	3
7	73	60	2.00	2
8	64	70	2.30	3
9	87	104	1.00	3
10	27	30	1.10	4
11	64	64	1.10	4
12	..[c]	...[c]	...[c]	2
13	77	79	2.00	3
14	..[c]	...[c]	1.60	3

[a] Column numbering continuous with Table 13.
[b] Approximate rating for hole filling: 1, best; and 4, least complete.
[c] Test not made; see text.

During the driving, the scale from eight rivets (the same eight for all steels) was caught and weighed. In Col. 29, Table 14, these weights are recorded on a relative basis, that for carbon steel being 1.00. After the driving, the packs were planed down to the center line of each of the two rows of rivets and ground lightly to remove burrs. The minute openings between the rivet shank and the edge of the hole were evaluated and recorded in thousandths of an inch. Averages were computed, and a rating as to the property of hole-filling assigned, as shown in Col. 30, Table 14. Although this work was performed objectively, it could not be expected that data of this sort could give other than very broad indications. All of the hole-filling appeared to be of an acceptable standard. Photographs of all of these machined packs are on record.

It will be seen from Cols. 29 and 30 that there is no correlation between amount of scale and filling of holes. If anything the tendency is in reverse; possibly the amount of scale is not so important as the condition that it be loose and easily knocked off, so that less of it is driven between the rivet and the hole.

It may be added that the test supervisor was requested to observe scaling particularly and the possible influence of scaling upon driving conditions, and to report any case in which he felt that scale was interfering with satisfactory use of the steel. His observation supplied nothing to be added to the "data" presented in these columns.

The former 3% nickel rivet, latterly unused, has previously been reported as unsatisfactory because of excessive scaling. In these tests it will be noted that Steels 8, 14, and 7, with nickel of 1.30, 1.88, and 1.96, respectively, seem neither better nor worse in scaling and hole-filling than Steel 6 with nickel as low as 0.10.

There seems to be nothing here that is tangible enough and detrimental enough to require recognition in the specification.

Loosening.—This phase of the investigation arose out of prior experience with silico-manganese rivets as previously reported by the writer to A. S. T. M. wherein rivets driven in shop and field, and originally found tight, were distinctly loose after a matter of days or weeks, the only remedy being additional original driving time.

Short-time driving tests (5 to 7 sec) were conducted on only a few of the steels. The number of rivets which were originally tight, and later loosened, was greater than practice could tolerate. This result showed that in all probability any high-strength rivet thus under-driven will soon become loose.

The specimen that was adopted for use in studying the degree of loosening after adequate driving comprised two plates 14 in. by $\frac{3}{8}$ in. by 3 ft $9\frac{1}{2}$ in., to which were added on each side two plates 7 in. by $\frac{5}{8}$ in. by 3 ft $9\frac{1}{2}$ in. These plates provided four rows of thirteen rivets each; namely, twenty-six rivets of $\frac{3}{4}$-in. grip and twenty-six rivets of $3\frac{1}{4}$-in. grip, for each rivet steel used. The plates were of high tensile steel, and it was soon found impossible to keep any of the outer, or shorter, rivets tight unless the inner rows were completely driven first.

After all rivets were driven for 24 to 32 sec, according to the "feel" of the job to the riveter, there was not any instance in which a rivet originally tight proved later to be loose.

Head Forging.—In this test, three 1-in. rivets of each steel were driven as usual until the point was formed into a good head; then this head was further driven down with a flat snap until it had a diameter of $2\frac{1}{2}$ in. Only four steels failed to survive this test with no evidence whatever of edge cracking. The two that showed definite cracking, Items 7 (0.14 C, 0.37 Mn, 1.96 Ni, 0.96 Cu) and 8 (0.11 C, 0.72 Mn, 1.30 Ni, 1.57 Cu), and one that showed traces of it, Item 9 (0.11 C, 0.74 Mn, 0.75 Ni, 1.28 Cu), have a marked resemblance in their high copper content. None of the steels cracked on the top of the flattened heads.

This edge cracking would probably be evidence of undesirability to some engineers, but it would not be so to a metallurgist. The phenomenon is expected with high copper content.[3] It is confined to non-forged edges and does

[3] "The Alloys of Iron and Copper," by J. L. Gregg and B. N. Daniloff, 1st Ed., Published for Eng. Foundation by McGraw-Hill Book Co., Inc., 1934, pp. 75–97.

not appear on forged surfaces, such as the normal rivet heads formed in the other series of tests. The facts are recorded for what they are worth, but all the steels in the tests are considered to have forged adequately for use as rivets, and no test of this nature was recommended to A. S. T. M. as a standard or acceptance test.

Conclusions

1. Probably any of the compositions stated in Table 11 which do not exceed the upper tensile limit of Specification A195 will make a satisfactory structural rivet.

2. It should be understood that any high-tensile rivet requires a longer driving time than a carbon steel rivet of the same size and under the same conditions, and that this is particularly true where the driving is in high-tensile material.

3. Before employing any high-strength composition of rivet steel other than silico-manganese, there should be made (if not previously made and results at hand):

(*a*) A test for shearing strength of driven rivets, or an air-blast quench test of the bar, or possibly both; and

(*b*) A driving test for tightness and for subsequent loosening.

4. Under Specification A195 there is a dependable static rivet shearing strength of 64,000 lb per sq in., based on the cold rivet diameter. This applies to double-shear testing, and the unit shearing strength of single-shear specimens would be higher, according to the records of other experimenters.

5. There is no reason to employ a stronger rivet than such as will qualify under Specification A195, if the parts to be riveted are of silicon (A94) or comparable steel.

6. A working unit shear of 20,000 to 22,000 lb per sq in. is in line with other features of standard specifications for static structures.

7. The question of fatigue strength is not included in this report.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MOMENT BALANCE: A SELF-CHECKING ANALYSIS OF RIGIDLY JOINTED FRAMES

Discussion

BY MESSRS. D. D. MATTHEWS, AND WILLIAM A. LARSEN

D. D. MATTHEWS,[5] JUN. AM. SOC. C. E.[5a]—The "moment-balance" method of analyzing statically indeterminate frames may deservedly claim attention for two reasons: First, it involves a principle that yields an interesting picture of the structural action involved. Any set of assumed joint moments is adjusted in the analysis to satisfy statics and the geometry of continuity. "Moment balance" is essentially an automatic trial and revision process. Secondly, the practical convenience in applying the method to highly indeterminate frames with negligible sidesway is sufficient to repay the time spent in mastering the method, because of the remarkable self-checking feature of the procedure. Figs. 4, 5, 6, and 7 illustrate an example of a moment-balance analysis of a three-story frame with fifteen joints. The tables in Fig. 7 differ from those in Mr. Cornish's example only in that, for ease of comparison, the columns for the first and second balances have been placed next to each other.

For example, consider joint 3, Fig. 5, as isolated in Fig. 6. The value of X in this case is $+ 81$ (or $- X = - 81$). The balanced moment, $- 19.1$ in Fig. 7, is computed from Eqs. 5 and 6, as follows: $\kappa = \frac{10}{30.5}(81 - 26.8) = + 17.8$; and $M = \frac{1}{2}(- 81 + 25 + 17.8) = - 19.1$. This is the first balance; the result of the second balance (see Fig. 7) is 19.8. The units throughout are in ton-feet, and the analysis neglects sidesway.

For a practical analysis, blanks containing the outlines of Figs. 4, 5, and 7

NOTE.—This paper by R. J. Cornish, Esq., appears on pp. 683–690 of this issue of *Proceedings*.

[5] Lt. Comdr., R. N. V. R., England.

[5a] Received by the Secretary April 25, 1942.

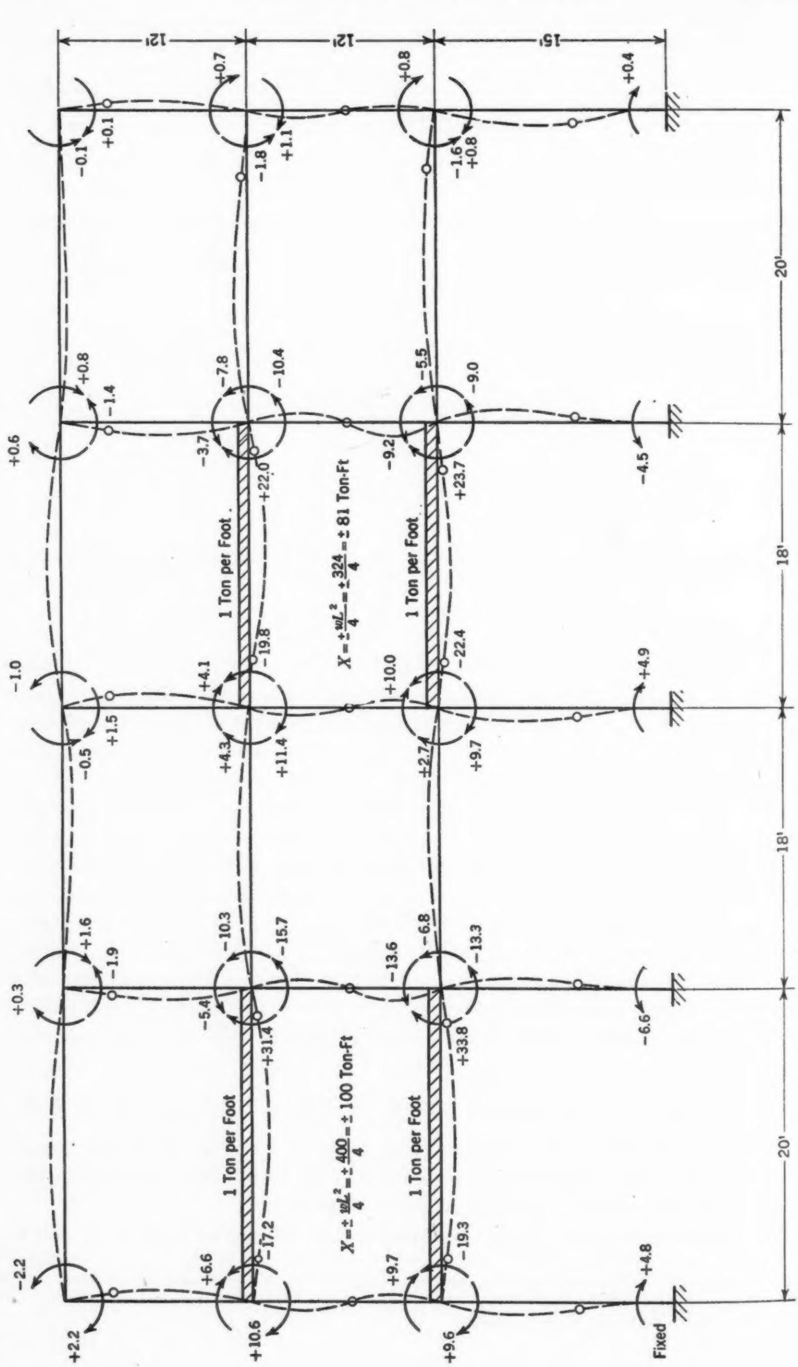

FIG. 4.—DIMENSIONS, LOADS, AND FINAL MOMENTS

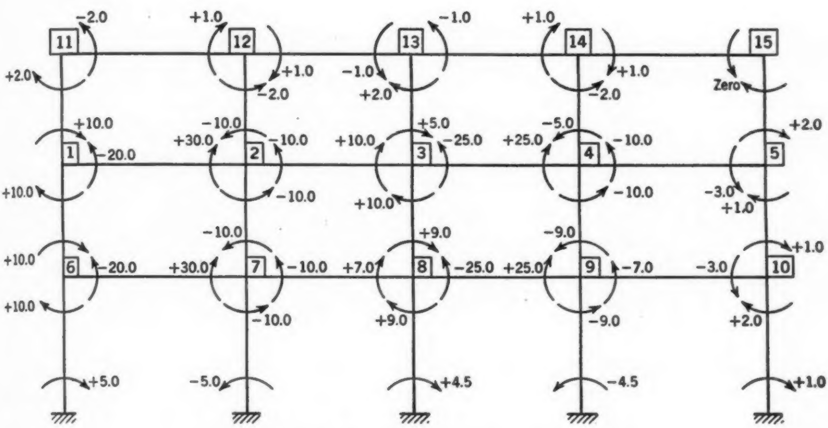

FIG. 5.—ASSUMED MOMENTS, AND ORDER OF BALANCING

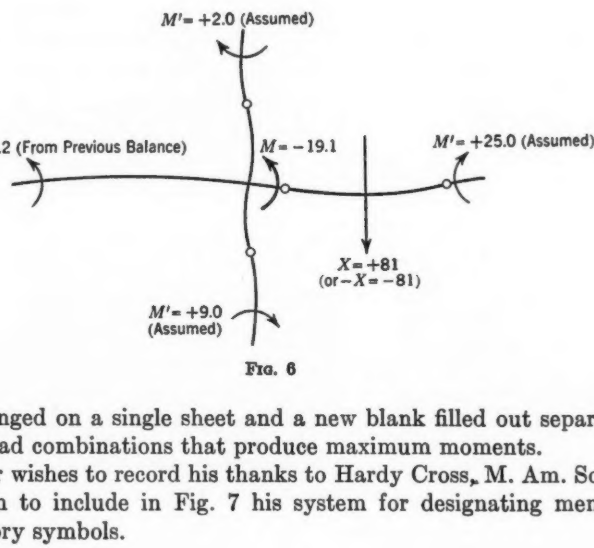

FIG. 6

could be arranged on a single sheet and a new blank filled out separately for each of the load combinations that produce maximum moments.

The writer wishes to record his thanks to Hardy Cross, M. Am. Soc. C. E., for permission to include in Fig. 7 his system for designating members by self-explanatory symbols.

MEMBER	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
K			2.5	2.5	6.0	6.0	8.5	8.5
−X			−	−	−	−	−	−
M'			+6.5	+6.6	+1.0	+0.5	+7.5	+7.1
κ			−2.2	−2.1	−5.3	−5.0	−7.5	−7.1
2M			+4.3	+4.5	−4.3	−4.5	−	−
M			+2.1	+2.2	−2.1	−2.2	−	−

6

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
6.0	6.0			3.5	3.5	8.0	8.0	17.5	17.5
−	−			−	−	−	−	−	−
−2.1	−2.2			−6.0	−5.4	−1.0	−0.6	−9.1	−8.2
+3.1	+2.8			+1.8	+1.6	+4.2	+3.8	+9.1	+8.2
+1.0	+0.6			−4.2	−3.8	+3.2	+3.2	−	−
+0.5	+0.3			−2.1	−1.9	+1.6	+1.6	−	−

8

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1
8.0	8.0			3.5
−	−			−
+1.6	+1.6			+4.1
−2.7	−2.6			−1.2
−1.1	−1.0			+2.9
−0.6	−0.5			+1.5

1st Balance
2nd Balance
1st Balance
2nd Balance

2.5

3.5

MEMBER	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
K	2.5	2.5	2.5	2.5	8.0	8.0	13.0	13.0
−X	−	−	−	−	−100	−100	−100	−100
M'	+2.0	+2.1	+10.0	+10.0	+30.0	+30.1	+42.0	+42.2
κ	+11.1	+11.1	+11.1	+11.1	+35.8	+35.6	+58.0	+57.8
2M	+13.1	+13.2	+21.1	+21.1	−34.2	−34.3	−	−
M	+6.5	+6.6	+10.6	+10.6	−17.1	−17.2		

8

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
8.0	8.0	3.5	3.5	7.0	7.0	10.0	10.0	28.5	28.5
+100	+100	−	−	−	−	−	−	+100	+100
−17.1	−17.2	−2.0	−2.1	−10.0	−13.8	+10.0	+4.3	−19.1	−28.8
−22.7	−20.0	−10.0	−8.7	−19.9	−17.5	−28.4	−25.0	−81.0	−71.2
+60.2	+62.8	−12.0	−10.8	−29.9	−31.3	−18.4	−20.7	−	−
+30.1	+31.4	−6.0	−5.4	−14.9	−15.7	−9.2	−10.2		

See Fig. 6 and Supporting Text

10

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1
10.0	10.0	3.5	3.5	7.0
−	−	−	−	−
−9.2	−10.3	+2.0	+1.5	+9.0
+17.8	+18.9	+6.2	+6.7	+12.4
+8.6	+8.6	+8.2	+8.2	+21.4
+4.3	+4.3	+4.1	+4.1	+10.7

2.5

7

MEMBER	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
K	2.5	2.5	4.0	4.0	8.0	8.0	14.5	14.5
−X	−	−	−	−	−100	−100	−100	−100
M'	+10.6	+10.6	+5.0	+5.0	+30.0	+32.8	+45.6	+48.4
κ	+9.4	+8.9	+15.0	+14.2	+30.0	+28.5	+54.4	+51.6
2M	+20.0	+19.5	+20.0	+19.2	−40.0	−38.7	−	−
M	+10.0	+9.7	+10.0	+9.6	−20.0	−19.3	−	−

8

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
8.0	8.0	7.0	7.0	12.0	12.0	10.0	10.0	37.0	37.0
+100	+100	−	−	−	−	−	−	+100	+100
−20.0	−19.3	−14.9	−15.7	−5.0	−6.7	+7.0	+3.1	−32.9	−38.6
−14.5	−13.2	+12.7	−11.6	−21.8	−19.9	−18.2	−16.6	−67.2	−61.4
+65.5	+67.5	−27.6	−27.3	−26.8	−26.6	−11.2	−13.5	−	−
+32.8	+33.8	−13.8	−13.6	−13.4	−13.3	−5.6	−6.8	−	−

10

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1
10.0	10.0	7.0	7.0	12.0
−	−	−	−	−
−5.6	−6.8	+10.7	+11.4	+4.5
+11.9	+12.2	+8.3	+8.6	+14.3
+6.3	+5.4	+19.0	+20.0	+18.8
+3.1	+2.7	+9.5	+10.0	+9.4

4

+5.0	+4.8

12

−6.7	−6.6

+4.7

Fig. 7.—Moment-Balance Des

o| 2	o— 1	o— 2	Σ 1	Σ 2
3.5	8.0	8.0	19.5	19.5
–	–	–	–	–
+1.1	+1.0	+0.7	+6.7	+6.4
–1.1	–2.7	–2.6	–6.6	–6.4
+3.0	–1.7	–1.9	–	–
+1.5	–0.9	–1.0		

(8)

—o 1	—o 2	|o 1	|o 2	o| 1	o| 2	o— 1	o— 2	Σ 1	Σ 2
8.0	8.0			3.5	3.5	6.0	6.0	17.5	17.5
–	–			–	–	–	–	–	–
–0.9	–1.0			–3.9	–3.7	–	–0.1	–4.8	–4.8
+2.2	+2.2			+1.0	+1.0	+1.6	+1.6	+4.8	–4.8
+1.3	+1.2			–2.9	–2.7	+1.6	+1.5	–	–
+0.7	+0.6			–1.5	–1.4	+0.8	+0.8	–	–

(6)

—o 1	—o 2	o| 1	o| 2	|o 1	|o 2	Σ 1	Σ 2	MEMBER
6.0	6.0	2.5	2.5			8.5	8.5	K
–		–	–			–	–	–X
+0.8		+0.7				+1.5		M'
–1.1		–0.4				–1.5		K
–0.3		+0.3						2M
–0.1		+0.1						M

(3.5) (2.5)

o| 2	o— 1	o— 2	Σ 1	Σ 2
7.0	10.0	10.0	30.5	30.5
–	–81	–81	–81	–81
+9.5	+25.0	+22.5	+26.8	+23.2
+3.2	+17.8	+18.9	+54.2	+57.7
+22.7	–38.2	–39.6		
+11.4	–19.1	–19.8		

See Fig. 6 and Supporting Text

(10)

—o 1	—o 2	|o 1	|o 2	o| 1	o| 2	o— 1	o— 2	Σ 1	Σ 2
10.0	10.0	3.5	3.5	7.0	7.0	8.0	8.0	28.5	28.5
+81	+81							+81	+81
–19.1	–19.8	–2.0	–1.5	–9.0	–9.1	–3.0	–1.9	–33.1	–32.3
–16.8	–17.1	–5.9	–6.0	–11.8	–12.0	–13.4	–13.7	–47.9	–48.7
+45.1	+44.1	–7.9	–7.5	–20.8	–21.1	–16.4	–15.6	–	–
+22.5	+22.0	–3.9	–3.7	–10.4	–10.5	–8.2	–7.8		

(8)

—o 1	—o 2	o| 1	o| 2	|o 1	|o 2	Σ 1	Σ 2	MEMBER
8.0	8.0	2.5	2.5	2.5	2.5	13.0	13.0	K
–	–	–	–	–	–	–	–	–X
–8.2	–7.8	+1.0	+0.9	–	+0.1	–7.2	–6.8	M'
+4.4	+4.2	+1.4	+1.3	+1.4	+1.3	+7.2	+6.8	K
–3.8	–3.6	+2.4	+2.2	+1.4	+1.4	–	–	2M
–1.9	–1.8	+1.2	+1.1	+0.7	+0.7	–	–	M

(7) (2.5)

o| 2	o— 1	o— 2	Σ 1	Σ 2
12.0	10.0	10.0	39.0	39.0
–	–81	–81	–81	–81
+4.7	+25.0	+23.9	+34.6	+33.2
+14.7	+11.9	+12.2	46.4	47.7
+9.4	–44.1	–44.9	–	–
+4.7	–22.0	–22.4	–	–

(10)

—o 1	—o 2	|o 1	|o 2	o| 1	o| 2	o— 1	o— 2	Σ 1	Σ 2
10.0	10.0	7.0	7.0	12.0	12.0	8.0	8.0	37.0	37.0
+81	+81	–	–	–	–	–	–	+81	+81
–22.0	–22.4	–10.4	–10.5	–4.5	–4.5	–3.0	–1.9	–39.9	–39.3
–11.2	–11.3	–7.8	–7.9	–13.3	–13.5	–8.9	–9.0	–41.2	–41.7
+47.8	+47.3	–18.2	–18.4	–17.8	–18.0	–11.9	–10.9	–	–
+23.9	+23.7	–9.1	–9.2	–8.9	–9.0	–6.0	–5.5	–	–

(8)

—o 1	—o 2	o| 1	o| 2	|o 1	|o 2	Σ 1	Σ 2	MEMBER
8.0	8.0	4.0	4.0	2.5	2.5	14.5	14.5	K
–	–	–	–	–	–	–	–	–X
–5.9	–5.5	+1.0	+0.5	+1.2	+0.9	–3.7	–4.1	M'
+2.0	+2.3	+1.0	+1.1	+0.6	+0.7	+3.6	+4.1	K
–3.9	–3.2	+2.0	+1.6	+1.8	+1.6	–	–	2M
–1.9	–1.6	+1.0	+0.8	+0.9	+0.8	–	–	M

(12) (4)

–4.5	–4.5

+0.5	+0.4

+4.9

…CE DESIG… A THREE-STORY FRAME

William A. Larsen,[6] Esq.[6a]—Much has been written the past few years on methods of obtaining moments in frames by a series of successive corrections. For any new presentation to be of value, a definitely more desirable feature than is found in the present methods should be given.

This paper presents a moment-distribution method which is a direct application of the simple slope-deflection equations. For the more or less simple frames involving no sidesway the method appears to have an advantage. It has been shown that the slope-deflection equations are convergent in character—that is, they may be solved by a series of successive corrections. The author presents the equations in such a way that the "rate of convergence" of the assumed moments to the actual moments appears to be rapid. This is the main feature of the paper. If it is shown definitely that the convergence is rapid even in case the frame has large differences in K-values for the different members, this method should be recommended. A quick check by the writer seemed to indicate that such is the case. This method does not, however, present tools enabling the designer to solve any new problems which are difficult now, or to solve frames more easily than by the regular moment-distribution methods. Its advantage will be in the total time required to work a frame. Where a frame will balance in 3 or 4 corrections by the regular Hardy Cross method, this method will have no apparent advantage.

That "the arithmetical work is almost self-checking," a very desirable feature, is obvious although it is not unique to this method. The other moment-distribution methods may be worked so as to be almost self-checking also. The point that any error that is introduced will be eliminated in subsequent balances is important.

It should be emphasized that this method is based on the slope-deflection equations from which the terms involving variable moments of inertia as well as the deflection of a joint (sidesway) have been omitted. This method in its present form is not applicable, therefore, to frames having haunched beams which are so common in practice today, and is, consequently, limited in application.

In the regular moment-distribution methods now used, correction for sidesway can be incorporated in the ordinary procedure. To some it will be a disadvantage to change from the author's method to some other to obtain sidesway corrections. There is extra merit to a procedure that covers both the regular moments and sidesway moments. Many designers are still looking for a method to analyze, quickly, unsymmetrical frames having members of variable moments of inertia and subjected to sidesway.

The author is to be commended on his clear and concise presentation.

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[6] Associate Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[6a] Received by the Secretary April 27, 1942.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

CLASSIFICATION OF IRRIGABLE LANDS

Discussion

By R. Earl Storie, Esq.

R. Earl Storie,[2] Esq.[2a]—In discussing this paper it can be stated that soil technologists are extremely pleased and happy that engineers and economists are getting down to the fundamentals of land classification—the soil. As Mr. Johnston has stated, the ability to produce depends to a large degree on the quality of the soil.

Many world travelers who have studied the historical development of irrigated agriculture have questioned its permanency. Was this decadence due to wars, the loss of soil fertility, a general lack of capacity of the land to produce, or what? An inadequate inventory of the land resources before development took place may have been responsible for much of this decadence. This should be a real challenge to those working in the land-classification field. There is more arable land in San Diego County, California, than water, and it is extremely important to select areas of land that can be benefited by, and can pay for, the water. Agriculturalists are very happy that Mr. Johnston and his associates are going into the classification of land so thoroughly, because certainly there is no question but that this work should be done in advance of any construction.

Consider the question of the so-called specialized soils—those that may produce a very few specialized crops. Often these specialized soils will not pay the carrying charges, and soil specialists certainly agree with the land classifiers of the Bureau of Reclamation that the soils which will grow many different kinds of crops are the ones that will stand the test of time. It is cause for satisfaction that more rigid standards have been set up and adhered to during the 15 years since about 1927. Had the standard of land classification been as rigid 30 years ago, the Nation certainly would have less irrigation projects that stand as tombstones to the memory of unwise irrigation planning.

To aid this work further, investigators at the University of California, in Berkeley, have been supplying the land classifiers with Natural Land Type

Note.—This paper by W. W. Johnston, Esq., appears on pp. 667–682 of this issue of *Proceedings*.

[2] In charge, Soil Survey, Univ. of California, Berkeley, Calif.

[2a] Received by the Secretary March 25, 1942.

Maps covering the various parts of the state. The natural land type is a land unit having more or less uniform, stable conditions of slope, soil profile, and soil texture, and the less stable factors of drainage, salt content, erosion, nutrient level, and micro-relief.

The soil profile is probably the one factor that is least considered by land classifiers, yet it probably is the most important over a long period of time. By all means it should be very carefully considered to insure a permanent irrigated agriculture.

There seems to be some danger in using a too generalized reconnaissance soil map for land-classification purposes. In the future there will be less danger because modern soil surveys are made on a scale sufficient to show factors of soil texture, profile, slope, erosion, alkali content, drainage, etc.

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DISCUSSIONS

NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS

Discussion

BY BRUCE JOHNSTON, ASSOC. M. AM. SOC. C. E.

BRUCE JOHNSTON,[19] ASSOC. M. AM. SOC. C. E.[19a]—The numerical procedure presented by Professor Newmark has advantages of simplicity, accuracy, and speed that make application to actual design work particularly effective. In an extension course given at Lehigh University, in Bethlehem, Pa., the writer has had the opportunity of presenting the method in detail to a number of engineers. Several of these engineers have found the procedure superior to other similar methods. The procedure was recently applied in connection with the analysis and design of several unusual mill building frames that are now (May, 1942) under construction.

As stated by the author (see "Synopsis"), "The essential features of the procedure are not new"—they are based on the well-known relations between the geometry of a bent beam and its moment-stiffness ratio. The importance of the procedure is not its newness, but its usability in actual design. It reduces the analysis of bending and buckling of struts to a systematic and accurate procedure of arithmetic, with a minimum chance of computational errors, and is exact enough for most applications. In actual structural members the moment of inertia frequently varies in a manner that makes actual integration of the fundamental differential equations exceedingly complex, if not impossible. Simple numerical procedures such as the author's deserve relatively more attention in structural engineering literature than they now have.

The practical usefulness of the procedure in continuous frame analysis will be increased if a summary is made of its relation to the slope-deflection and moment-distribution procedures for obtaining terminal moments of members in framed structures. In Fig. 21 is shown a rotation notation for the angle changes due to unit positive moments applied at either end of a simply supported member. Moments are assumed as positive when they apply a clock-

NOTE.—This paper by N. M. Newmark, Assoc. M. Am. Soc. C. E., appears on pp. 691–718 of this issue of *Proceedings*.

[19] Associate Director, Fritz Eng. Laboratory, and Associate Prof. of Civ. Eng., Lehigh Univ., Bethlehem, Pa.

[19a] Received by the Secretary April 15, 1942.

wise couple to the end of the beam. The angles of rotation of the end tangents of the beam axis are also considered positive when clockwise. The first subscript indicates the location of the angle change and the second subscript indicates the location of the applied unit moment—that is, ϕ_{AB} = angle change at A due to the unit moment at B.

By the law of reciprocal deflections, $\phi_{AB} = \phi_{BA}$. The three independent angle changes ϕ_{AA}, ϕ_{BB}, and ϕ_{AB} may be determined by model analysis or by two applications of the simple numerical procedure described by the author and illustrated in Figs. 10 and 11. In the symmetrical member $\phi_{AA} = \phi_{BB}$, and only one application would be necessary. The angle changes ϕ'_A and ϕ'_B, due to any applied load (also shown in Fig. 21), are determined easily by

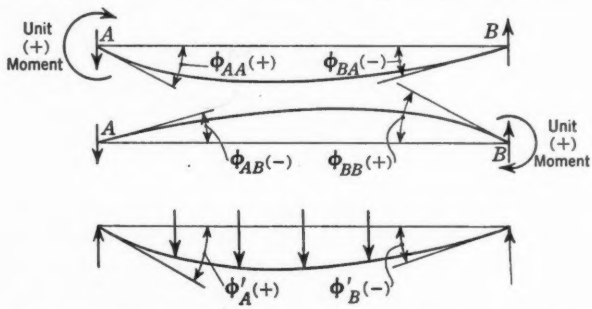


FIG. 21.—END-ANGLE CHANGES DETERMINED BY NUMERICAL PROCEDURE

one additional application of the author's numerical procedure. Note the difference in the sign convention for the terminal moments M_A and M_B; but the sign of the angle changes will be the same as that of the end slopes in the author's paper. These five angle changes determine any or all of the coefficients needed in a generalized solution either by slope deflection or moment distribution. The effect of direct load upon the bending stiffness could be included, but is usually neglected in bridge and building frame analysis.

The positive rotation notation for moments, shears, angle changes, and lateral translation of the ends of any member in a loaded frame is shown in

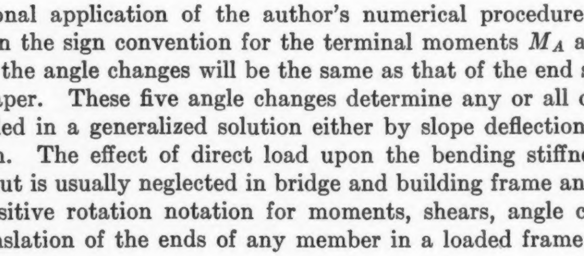


FIG. 22.—MOMENTS, SHEARS, ANGLE CHANGES, AND LATERAL TRANSLATION OF ANY FRAMED MEMBER, SHOWN AS POSITIVE

Fig. 22. In the case of the member with uniform cross section, the "slope-deflection" equations are written:

$$M_{AB} = \frac{2\,E\,I}{l}\left(2\,\theta_A + \theta_B - \frac{3\,\Delta}{l}\right) \pm M_{FA} \qquad \text{(14a)}$$

and

$$M_{BA} = \frac{2\,E\,I}{l}\left(2\,\theta_B + \theta_A - \frac{3\,\Delta}{l}\right) \pm M_{FB} \ldots\ldots\ldots\ldots (14b)$$

in which M_{FA} and M_{FB} are "fixed-end" moments due to loads on the beam span. For downward loads on a horizontal member, M_{FA} is negative and M_{FB} is positive.

It may be shown by the "moment-area" relations that the following slope-deflection equations obtain for the general case of variable I, written in terms of the fundamental angle changes shown in Fig. 21:

$$M_{AB} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}}\left[\phi_{BB}\,\theta_A - \phi_{AB}\,\theta_B + (\phi_{AB} - \phi_{BB})\,\frac{\Delta}{l} + \phi_{AB}\,\phi'_B - \phi_{BB}\,\phi'_A\right] \ldots\ldots\ldots\ldots (15a)$$

and

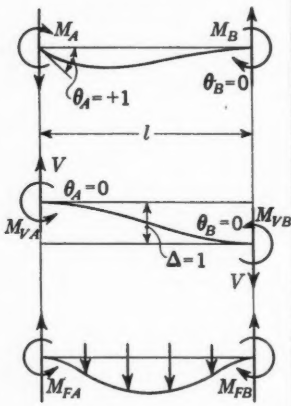
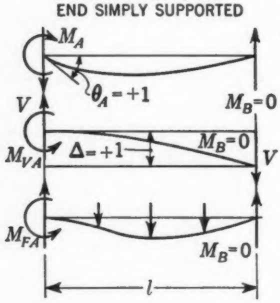
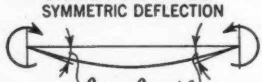
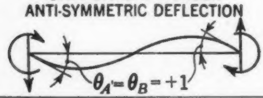
$$M_{BA} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}}\left[\phi_{AA}\,\theta_B - \phi_{AB}\,\theta_A + (\phi_{AB} - \phi_{AA})\,\frac{\Delta}{l} + \phi_{AB}\,\phi'_A - \phi_{AA}\,\phi'_B\right] \ldots\ldots\ldots\ldots (15b)$$

The factors commonly used in the moment-distribution procedure, as defined by the general slope-deflection equations (Eq. 15), are given in Table 1.

In the actual calculation of the basic end-angle changes ϕ_{AA}, ϕ_{BB}, ϕ_{AB}, ϕ'_A, and ϕ'_B, the procedure used by the author in Figs. 10 and 11 may be simplified. Deflections need not be calculated at all. The end slopes are simply equal to the "conjugate" beam-end reactions caused by the "equivalent concentrated angle changes" treated as loads. Upward end reaction is positive on the left end and negative on the right end. The equivalent concentrated angle-change loads should be calculated by the formulas given in Figs. 3 and 5, and the alternate to the author's procedure in Fig. 10 is presented in Fig. 23. The results check those of Fig. 10. If a mechanical calculating machine is used in the computation, it would be convenient to divide the member into either five or ten segments. The calculation of end reactions is then obtained by multiplying the equivalent concentrated angle-change loads by successive decimal fractions, 0.2, 0.4, etc., or 0.1, 0.2, 0.3, etc., up to 1.0, in the case of five or ten segments, respectively.

The other angle changes ϕ_{BB}, ϕ'_A, and ϕ'_B (also a check on ϕ_{AB}) are obtained by two additional sets of computation similar to Fig. 23. It should be noted that in computing ϕ_{BB} and ϕ_{AB} ($= \phi_{BA}$) the signs of the results and the sense of the applied moment at the right end of the beam will be reversed from that shown in Fig. 11 to conform to the rotation sign convention used in this discussion. After the basic angle changes are obtained, they may be substituted in Eq. 15 or in Table 1 to provide the necessary basis for analysis either by the slope-deflection or moment-distribution procedures, respectively. These details will be obvious to one already familiar with structural frame analysis.

TABLE 1.—MOMENT DISTRIBUTION FACTORS FOR END A OF ANY MEMBER AB

				UNIFORM SECTION
STANDARD CASE - FAR END HELD FIXED 	CARRY-OVER FACTOR	$r_{AB}=\left\|\frac{M_B}{M_A}\right\| \begin{array}{l}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{array}$	$r_{AB}=-\frac{\phi_{AB}}{\phi_{BB}}$	$\frac{1}{2}$
	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{array}{l}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{array}$	$S_{MAB}=\frac{1}{\phi_{AA}+r_{AB}\phi_{AB}}$	$\frac{4EI}{l}$
	SHEAR STIFFNESS	$S_{VAB}=\left\|V\right\| \begin{array}{l}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{array}$	$S_{VAB}=\frac{1+2r_{AB}+\frac{r_{AB}}{r_{BA}}}{l^2(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{12EI}{l^3}$
	MOMENT DUE TO UNIT SIDESWAY	$M_{VA}=\left\|M_A\right\| \begin{array}{l}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{array}$	$M_{VA}=\frac{-(1+r_{AB})}{l(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{6EI}{l^2}$
Note - Factors at End B May Be Obtained by Interchanging Subscripts A and B	FIXED-END MOMENT	$M_{FA}=\left\|M_A\right\| \begin{array}{l}\theta_A=0\\ \theta_B=0\\ \Delta=0\end{array}$	$M_{FA}=\frac{-(r_{AB}\phi'_B+\phi'_A)}{\phi_{AA}+r_{AB}\phi_{AB}}$	$-\frac{4EI}{l}\times\left(\phi'_A+\frac{\phi'_B}{2}\right)$
SPECIAL CASE, FAR END SIMPLY SUPPORTED 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{array}{l}\theta_A=+1\\ M_B=0\\ \Delta=0\end{array}$	$S_{MAB}=\frac{1}{\phi_{AA}}$	$\frac{3EI}{l}$
	SHEAR STIFFNESS	$S_{VAB}=\left\|\frac{-M_A}{l}\right\| \begin{array}{l}\theta_A=0\\ M_B=0\\ \Delta=+1\end{array}$	$S_{VAB}=\frac{1}{\phi_{AA}l^2}$	$\frac{3EI}{l^3}$
	FIXED-END MOMENT	$M_{FA}=\left\|M_A\right\| \begin{array}{l}\theta_A=0\\ M_B=0\\ \Delta=0\end{array}$	$M_{FA}=\frac{-\phi'_A}{\phi_{AA}}$	$-\frac{3EI\phi'}{l}$
SYMMETRICAL MEMBER SYMMETRIC DEFLECTION 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{array}{l}\theta_A=+1\\ \theta_B=-1\\ \Delta=0\end{array}$	$S_{MAB}=\frac{1-r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{2EI}{l}$
SYMMETRICAL MEMBER ANTI-SYMMETRIC DEFLECTION 	MOMENT STIFFNESS	$S_{MAB}=\left\|M_A\right\| \begin{array}{l}\theta_A=+1\\ \theta_B=+1\\ \Delta=0\end{array}$	$S_{MAB}=\frac{1+r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{6EI}{l}$

The writer has discussed the application of the paper to structural frame analysis in cases where direct stress in a member may be neglected in so far as its effect on bending is concerned. The author's procedure is particularly adapted to the computation of critical buckling loads under direct stress for

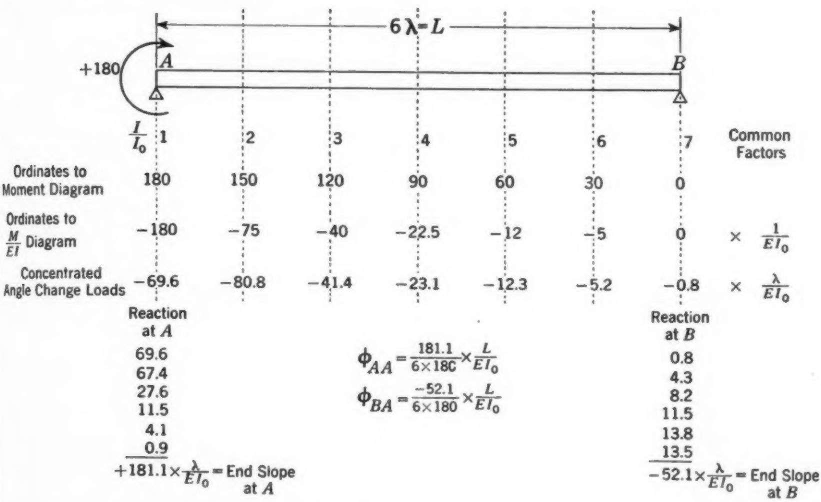


Fig. 23.—Alternate Procedure to Fig. 10 for Calculation of End Slopes Only

cases of nonuniform cross section. The method is clearly outlined by the author. The procedure furnishes the engineer with a simple method involving only the processes of arithmetic, and thereby bears a relation to the elastic stability theory of bars similar to that which the Hardy Cross moment-distribution method bears to older and more cumbersome methods of structural frame analysis.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

PILE-DRIVING FORMULAS

PROGRESS REPORT OF THE COMMITTEE ON THE BEARING VALUE OF PILE FOUNDATIONS

Discussion

By R. E. Bakenhus, M. Am. Soc. C. E.

R. E. Bakenhus,[75] M. Am. Soc. C. E.[75a]—The Committee was originally designated as the "Committee on Pile Driving Formulae and Tests."[76] Fortunately the title was changed, for up to this point "pile formulas" is the one subject upon which the Committee has reached no definite stated conclusions. It is to be noted that the subject of pile formulas is only one chapter of twelve in the proposed Manual on the Bearing Value of Pile Foundations, which the Committee has nearly completed. The present Report is not all the Committee has to offer, as was assumed by some discussers. There is much to be considered besides the safe bearing value of an individual pile. Publication of the Progress Report on Pile-Driving Formulas has resulted in illuminating and profitable discussion that has presented many points of view of this controversial subject. There should be nothing really controversial in engineering; when the term is applied it may indicate mental confusion on a subject that is not fully understood or regarding which there is insufficient knowledge.

Many engineers never have need of pile foundations in their practice; others use pile foundations occasionally and cannot be fully versed in the intricacies of the foundation problem, whereas a few are so heavily engaged in these problems and in decisions as to piles that their experience is a valuable guide. The two latter groups would look upon a manual that included pile-driving formulas from different points of view.

Note.—This Report was published in May, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1941, by Messrs. G. G. Greulich, C. O. Emerson and D. O. Northrup, Harry J. Engel, and John D. Watson; October, 1941, by Messrs. Robert D. Chellis, Lazarus White, John G. Mason, Carlton S. Proctor, George Paaswell, and Abraham Woolf; November, 1941, by Messrs. Howard T. Evans, William G. Atwood, Donald M. Burmister, Wallace E. Belcher, Clement C. Williams, and D. P. Krynine; December, 1941, by Messrs. Trent R. Dames and William W. Moore, Maxwell M. Upson, Gregory P. Tschebotarioff, Robert F. Legget, and Jacob Field; January, 1942, by Messrs. Lewis C. Wilcoxen, H. A. Mohr, and A. E. Cummings; February, 1942, by Messrs. Karl Terzaghi, Ralph B. Peck, and Arthur Casagrande; and March, 1942, by C. W. Dunham, M. Am. Soc. C. E.

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[75a] Received by the Secretary April 8, 1942.

[76] *Year Book*, Am. Soc. C. E., Vol. 61, 1935, p. 16.

A typical point of view is presented to the Committee in a resolution of the Mississippi Valley Conference of State Highway Departments, which reads as follows:

> "Be it resolved that it is the sense of the Mississippi Valley Conference of State Highway Departments, that public road bridges should be constructed under a method of control whereby pile driving may be done under a specification permitting the use of a dynamic driving formula for determining the safe bearing capacity of individual piles. This position is taken because of the economical limitations which are set up in highway construction where only a small number of piles are used in each of the many remote and scattered locations within the jurisdiction of any one Engineering organization.
>
> "Be it further resolved that a copy of this resolution be forwarded to the Committee on Bearing Value of Pile Foundations, American Society of Civil Engineers."

Another point of view is developed in the case of the engineer who is constantly engaged on extensive foundation undertakings, who has actual experience with pile driving in numerous localities, and who has at hand the facilities and means for complete soil, driving, and loading tests, which are in many cases denied to the other group.

Both groups must reach a decision that first results in a safe foundation, and second avoids waste or unnecessary expenditure. Tests cost relatively little in extensive operations, but may be relatively large and even out of the question with the smaller project. At its best, the pile-driving formula is merely an empirical method for predicting the safe bearing load for a single pile as determined from the weight and drop of the hammer and the penetration of the pile per blow.

The statement may be made unequivocally that the bearing value of a single pile, even if tested with a load, may be no indication of bearing value of a pile foundation as a whole, since failure may result from causes that cannot possibly be disclosed by testing a single pile. This is all discussed in other parts of the Manual, soon to be offered for publication. Experience has shown that there is no determinable fixed relation between the safe bearing value of a pile and the factors used in the formula. It is, therefore, a dangerous proceeding for an engineer to design or build a pile foundation solely on the information obtained by the usual test of measuring penetration per blow, height of fall, and weight of hammer. Some of these dangers may be eliminated by using an excessive factor of safety, in which case foundation costs may increase. This may be justified in some cases.

In the course of the remarks by thirty discussers, there has been some criticism. The membership of the Committee is really excellent, and consists of a representative cross section of the Society membership. Fortunately, experts and general practitioners are both represented, all with experience in piles. The General Chairman must take upon himself the unfavorable remarks that have been made in the discussion, and cannot agree to having them transmitted to the Committee membership. Some of the criticism, implying a lack of knowledge or vision on the part of the Committee, is based on a misconcep-

tion by the respective discussers, who were not aware of the existence of eleven other chapters, which unfortunately could not be published at this time. Some of the discussers apparently did not read the "Foreword"[77] with care, because some of the points are covered there.

The problem of the Committee has been a difficult one. Admittedly, pile formulas are in a controversial position not only among the discussers, but among the members of the Committee itself. It is the belief of the writer that such a situation should be handled by gentle methods, involving dissemination of knowledge and discussion of points of view—an educative method, in short—and that it should not be handled by the method of a dictator laying down the law as to what is right and what is wrong.

The writer is the first to recognize and to be critical of the shortcomings of the Chairman. In extenuation it can only be said that the bringing together of points of view of a large, diversified membership has taken much time and patience. It has required much patience on the part of the members, particularly in awaiting results of their work. Many of the Committee have been on urgent work, some of it abroad.

The writer was perhaps one of the first to point out that a building foundation was not necessarily safe because each one of the piles was tested by the hammer or by actual load. During the World War I, out of some 7,000 buildings coming under the cognizance of the writer as "second in command," one building—a large six-story warehouse 180 ft by 200 ft—constructed on pile foundations under urgent war-pressure conditions began to settle badly in a year or two (about 1920) until the contours of settlement showed some 16 in. maximum settlement slightly off center of the middle of the building, with diminishing settlement to a minimum of about 2 in. or 4 in. along the high edge. Later investigation showed that the piles had been driven into a hard material capable of sustaining the individual pile load, but the site was underlain by a compressible foundation that yielded to the weight of the building. Underpinning operations to satisfactory bottom were successfully undertaken. In two others of the 7,000 buildings, somewhat similar settlement occurred. This happened during the period 1918 to 1921, and the writer well remembers discussing this phenomenon with the late Daniel E. Moran, M. Am. Soc. C. E., one of the early foundation experts.

Mr. Chellis offers[78] a thorough discussion of the pile-formula problem. The principal conclusion to be drawn from his discussion is that the engineer in designing and building pile foundations must depend to a great extent on experience and judgment, both in securing reliable information in regard to the site and in choosing the type and weight of the pile and hammer to be used in driving them.

Some of his discussion covers ground contained in the data for the proposed Manual. Mr. Chellis comes to the conclusion that the "Engineering News" formula, Eq. 6, is not a general answer to the problem, although there are cases in which it is suitable for use. He points out that, in the case of a steel pile shell driven before filling with concrete and without a mandrel, as com-

[77] *Proceedings*, Am. Soc. C. E., May, 1941, p. 853.

[78] *Ibid.*, October, 1941, p. 1517.

pared with the driving of a steel shell containing a heavy steel driving mandrel, the empty shell by the "Engineering News" formula requires, in certain cases, some 50% greater penetration under identical conditions. Evidently this is due in large part to the fact that the "Engineering News" formula takes no special account of the relative weights of the two driven piles; naturally it does not, since it was devised in the time when only wooden piles were commonly used.

He also points out that a formula, even if it takes account of all energy losses, has limitations due to: The non-validity of a dynamic formula when driving in cohesive soils; lack of relationship between individual pile value and that of a group, involving soil capacity under the structure as a whole; reduction in useful driving energy at the tip of the pile, extending to complete disappearance of net driving energy after deducting losses; double-acting steam hammers operating under reduced driving pressure; reduced driving energy under water and when driving batter piles; and variations in efficiency of hammer. Mr. Chellis gives an analysis of the efficiency of different types of hammer, based on his experience. He recommends a simple field formula which the engineer who knows the field conditions and factors entering into the problem has worked out for the special situation encountered by the field inspector.

He also gives a sample computation for penetrations and applies them to four types of cases. The results are illuminating. Much of his comment and information should be incorporated into the Manual. He analyzes these cases by the use of Eqs. 33 and 35; in the latter the dynamic resistance of the pile appears as a result of deducting four quantities (experimentally or theoretically determined from the conditions of pile driving) from the quantity representing the total of dynamic force applied. The examples given show rather satisfactory results when piles of different weight and different types are used, as well as when different hammers are used for the same pile.

Mr. White[79] discourages the use of formulas, and emphasizes that he has repeatedly compared results of the "Engineering News" formula with actual tests on piles hydraulically loaded and has found the widest variations. All must agree with his statement that "it would be a calamity for the Society to lend its authority to the promulgation of any pile-driving formula as yet described," since it has been the universal experience that any given formula does not apply to all conditions. To blindly depend upon any formula would be a matter of guesswork, and might lead to extravagance or even disastrous results. Certainly the Society must emphasize that, if formulas are used, they must be used with judgment and discretion, taking into account the soil and driving conditions in every particular case.

Mr. White further refers to the difficulty in deciding whether or not to use piles in a foundation. This phase of the Committee problem is not covered in the Progress Report on Pile-Driving Formulas, since the subject matter belongs in other parts of the Manual. Certainly the Society should not take action adverse to what would be considered wise from the viewpoint of long practical experience. The problem is not a simple one when it is realized that many engineers of far less experience occasionally must use piles, and they have a

[79] *Proceedings*, Am. Soc. C. E., October, 1941, p. 1538.

right to look to the proposed Manual for a certain amount of guidance and instruction.

Mr. Mason[80] offers a comprehensive discussion. He concludes that pile-driving formulas are a necessity. He proposes a pile-driving formula of simple structure in which the over-all relation of the dynamic effect of the falling hammer without deduction of any losses is compared with a load test of the pile to failure. No description of the means of testing the pile to failure, or what is considered failure, is given. However, tables are shown for (*a*) steel piles driven by drop hammers; (*b*) steel piles driven by steam hammers; (*c*) solid concrete piles driven by steam hammers; and (*d*) timber piles driven by steam hammers. In some fifty-three cases the weight of the hammer, height of fall, and penetration are given, together with the dynamic force of the hammer and the test load to failure. The test load divided by the total dynamic force of the hammer gives the efficiency. Presumably when a pile foundation is to be used, it is the intention to drive test piles and load them to failure, thus obtaining the efficiency factor. From this the safe load of a pile may be determined, and the required penetration per blow to give a safe load may be then used with further piles.

The data from the tables have been recorded in Fig. 16, which shows as ordinates the formula load, and as abscissas the test load. The efficiencies are shown by lines radiating from the origin. The data show a rather wide distribution both in total load and in efficiency. On the right, the distribution of the tests in efficiency percentage groups is given, showing that most of them cluster between the 20% and 40% lines, and showing also a rather wide distribution in efficiency percentage. The method proposed should be useful under conditions where ample experience is available.

Mr. Proctor states[81] that the Progress Report will be of value if it succeeds in arousing full discussion, including the "'actual data determined from pile driving, application of formulas, and load tests * * *' as requested in the last paragraph of the 'Foreword.'" Fortunately, the discussion has been very satisfactory, although the amount of data submitted has been limited. The remarks by Mr. Proctor are basic, and should be followed by the Committee in work on the Manual.

Mr. Paaswell raises some pertinent points[82] that have been largely covered in the unpublished chapters of the proposed Manual. Whether the bearing value of a single pile is determined by formula, by load test, or by "pre-test," it is, of course, true that the sum of such load determinations for individual piles of a foundation does not necessarily give the bearing value of the entire foundation.

Mr. Woolf[83] discusses the formula problem and its change since the early "Engineering News" formula days, when wood piles were the only kind in use as compared with the heavy concrete piles or mandrel-driven steel shell piles of the present time. Mr. Woolf states that the load test is a dynamic test,

[80] *Proceedings*, Am. Soc. C. E., October, 1941, p. 1541.
[81] *Ibid.*, p. 1544.
[82] *Ibid.*, p. 1545.
[83] *Ibid.*, p. 1547.

which, of course, is true as long as the load causes movement of the pile in the soil. The static test, which he mentions as out of the question for any given pile, should be applied by careful long-time observations of the finished structure. He outlines a good method for approaching the foundation problem.

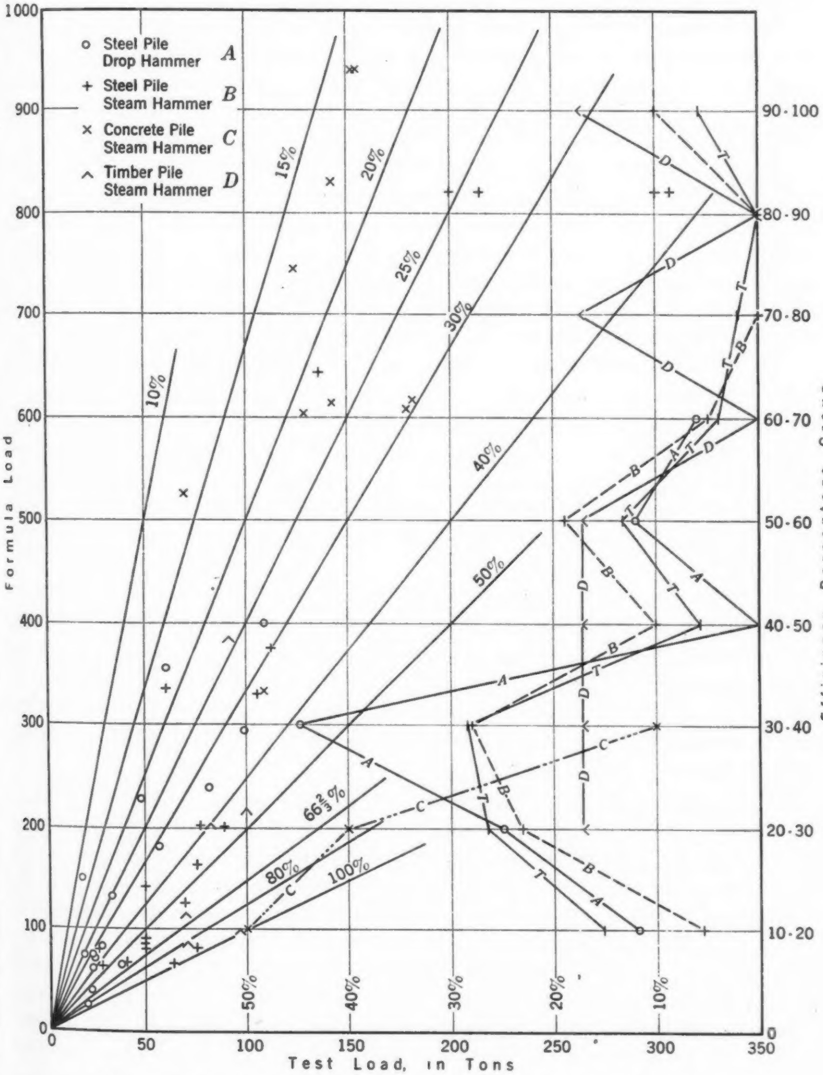

FIG. 16.—COMPARISON OF FORMULA LOADS AND TEST LOADS; MASON DATA

Mr. Greulich[84] states that it does not make much difference what formula is used as long as all data are observed carefully. During driving of test piles

[84] *Proceedings*, Am. Soc. C. E., September, 1941, p. 1391.

and after loading them to failure, the engineer puts suitable factors into whatever formula is chosen so that it is correlated to the test results. This is the empirical rather than the theoretical approach to the problem, and the method is successful if followed with care by an engineer of experience and good judgment.

The tables, which he gives, of ultimate loads by various formulas, compared with the actual test loads at failure, are very illuminating and show the futility of the blind use of any pile-driving formula. Mr. Greulich recommends placing some of the material of Report *A* in an appendix to the proposed Manual, and this is concurred in by some of the other discussers.

Messrs. Emerson and Northrup discuss[85] various formulas and point out one (Eq. 29) which will furnish the most convenient solution, although rather complicated for the average engineer on small projects. They state that the constant h_0 (the greatest hammer drop that will result in zero penetration) cannot be determined readily with a steam hammer, whether single or double acting. Table 4 shows variations in computed ultimate loads, for the purpose of demonstrating the influence of the various constants under discussion.

Mr. Engel gives[86] typical driving records of timber friction piles in New Orleans, La., bridge approaches. These records show the effect of a rest period between two similar tests on any given pile. They demonstrate that the suggested rule of a total net settlement of 0.01 in. for each ton of applied load may give satisfactory results when applied to point-bearing piles, where grouping and skin friction play a relatively small part, but that with friction piles all load tests should be made to complete failure in order to determine the ultimate skin friction.

Professor Watson favors Report *B* and points out[87] the futility of formulas in certain types of engineering work. He deplores the "moribund attitude" that prompted the preparation of Report *A* on Pile Formulas. However, he does not suggest what the engineer in the Midwest prairies should do when he has a total of perhaps twelve piles under some bridge foundation, and when neither funds nor time permit load tests or soil analysis. This is one of the difficult problems before the Committee.

Mr. Evans has developed a method[88] of applying a pile-driving formula on the job, which has proved itself in practice. He suggests that the pile itself can be regarded as a spring balance to arrive at the effect of driving at the tip of the pile. He shows curves of the elastic compression of the pile and the penetration per blow, and then uses this information in arriving at a check on the allowable load. He approaches the problem by preparing a plot in advance of the driving, showing a separate curve for each type and length of pile and for each kind of hammer used on the job. The curve shows the tip resistance to be expected for the entire range of values of the penetration, *s*. He protests against the development or use of any pile-driving formula as such, pointing out that it is misleading and unsafe to seek a magic combination of terms in a

[85] *Proceedings*, Am. Soc. C. E., September, 1941, p. 1396.
[86] *Ibid.*, p. 1398.
[87] *Ibid.*, p. 1400.
[88] *Ibid.*, November, 1941, p. 1784.

formula that will fit any and all cases, regardless, and which is supposed to indicate just what load a pile will support. He claims that instead of this the engineer should seek to promulgate the method of analysis. The "complete formula" should be applied not as a mystic and marvelous combination of terms, but as a series of steps in a logical method of analysis.

Colonel Atwood states[89] that there are no formulas of general or even local value, unless they are treated with good judgment and corroborated by many tests, and he inquires why, if that is true, one should try to use a formula. He gives two examples to bear out very clearly his point of view.

Professor Burmister states[90] that "* * * not enough is known, definitely, about the dynamics of pile driving and the way a pile develops its permanent bearing capacity for different subsurface conditions to warrant the inclusion of formulas in the Manual of Engineering Practice at the present time." He points out that "bearing capacity must be determined in relation to some maximum allowable safe settlement for a given structure." He observes that it does not seem likely that piles driven into soils of varying subsurface character will stress the soil mass under a static loading in anything like the same manner that the driving of a pile under the impact of a hammer will.

Mr. Belcher states[91] that the attempt to introduce a new formula is of doubtful value as it is based on the same fundamental data that invalidate the "Engineering News" formula. He mentions a case in which piles driven on the shores of the Hudson River showed 1 in. of penetration under seven blows of a double-acting steam hammer, whereas on the following day five blows were sufficient. He cites other cases in which the formula load increased after redriving.

President Williams notes[92] the underlying principles or spirit that governed the Committee in its work, and he has added valuable comment that should, in effect, be incorporated in the Manual.

Professor Krynine prefers Report *B* but states[93] that it is too concise and that its practical application will be difficult for an engineer not familiar with the bibliography of the subject. He states some necessary facts and principles that must be followed in applying any pile formula, and remarks that all the questions which he raises should be clarified in the Manual if, unfortunately, the Hiley formula is recommended for general use. He sincerely hopes, however, that this will not happen. He points out that empiricism is necessary in the application of formulas to given sites.

Messrs. Dames and Moore state[94] their belief that "* * * the viewpoint and emphasis of Report *B* represent more nearly a proper evaluation of the relative importance of dynamic pile formulas, static pile formulas, and pile loading tests than those presented in Report *A*." They also emphasize the difficulty of determining h_0 in the case of single-acting steam hammers (this applies also

[89] *Proceedings*, Am. Soc. C. E., November, 1941, p. 1789.
[90] *Ibid.*, p. 1790.
[91] *Ibid.*, p. 1791.
[92] *Ibid.*, p. 1793.
[93] *Ibid.*, p. 1794.
[94] *Ibid.*, December, 1941, p. 1939.

to double-acting steam hammers). They present results of extensive experience in the form of diagrams. Tests of certain piles demonstrate that the load which caused a permanent settlement of 0.01 in. per ton (proposed in the Progress Report) is far in excess of the safe bearing capacity or yield-point strength of any of these piles.

Mr. Upson states[95] "It would seem that, in order to attain this end [to provide a Manual of Engineering Practice for engineers engaged in pile driving] effectively, the simplest possible formulas and information should be advocated." He adds that Report *A* seems to have deviated widely from this principle and that he would eliminate most of the equations in it. He comments that "Report *A* indicates to the average reader that there is a desire to 'sell' Eq. 9 to the engineering profession, whereas it is well known that the 'Engineering News' formula (Eq. 6) is the most widely used and generally accepted." He points out some inconsistencies in the analysis of Report *A*, and is convinced that pile driving is by no means an exact science. Probably every one will agree with this conclusion. He states that his experience has developed almost no instances of failure of piles driven to the requirements of the "Engineering News" formula, in which good common-sense methods have been used, and that the few settlements which have occurred were due to other factors such as piles of inadequate structural rigidity or strength, lack of borings, excessive loading of the topsoil, the draining away or loosening of the penetrated material by adjacent operations, or the complete ignorance or omission of soil analysis.

Professor Tschebotarioff offers[96] a basic analysis of the Committee Report, pointing out that in its present form a compromise has been offered, although separate reports have been submitted. He prefers the viewpoint of Report *B* to that of Report *A*. He wisely states that "The importance of precise instructions concerning details of pile-testing procedures cannot be overemphasized." He points to a danger resulting from the fact that most present static formulas arbitrarily deal with friction, cohesion, and point-bearing resistance, which are assumed to have fixed values for a definite type of soil, whereas actually these formulas do not attempt to relate these values to the complicated shearing and compressive deformations in the soil around and beneath the pile.

Professor Legget[97] welcomes the later section of Report *B* as a useful outline of procedure for testing a bearing pile and points out that it could be expanded profitably before incorporation in the Manual. He advises that more attention be given to the interpretation of the results of loading tests. He regards pile formulas as of relatively minor importance in the design of pile foundations as a whole, and points out that "* * * this section of the Report [Report *A*] serves merely to bolster up the false importance so generally attached to pile formulas." He agrees with the original decision of the Committee, in expressing

> "* * * the hope that, as a result of the public discussion of the dual Report, agreement may be reached in the near future by the relegation of

[95] *Proceedings*, Am. Soc. C. E., December, 1941, p. 1947.
[96] *Ibid.*, p. 1949.
[97] *Ibid.*, p. 1951.

pile-driving formulas to their proper place as useful, but very limited, calculating aids in the construction of some pile foundations, after it has been found that bearing piles can be, and should be, usefully employed as foundation elements."

After discussing[98] the characteristics of the soil in design, Mr. Feld states that he

> "* * * would prefer to have the Manual covering pile-driving formulas include a definite formula for granular soils, a definite formula for plastic soils, and a definite formula for such conditions as end-bearing piles in which no lateral restraint or resistance is to be expected."

He adds that dynamic load tests are useless in plastic soils and that static load tests will give some dangerous results, even if conducted over a period of time, in plastic soils.

Mr. Wilcoxen offers[99] a record of an interesting formula that he developed as a result of small-scale tests.

Mr. Mohr states[100] that it is his "firm conviction that their inclusion [formulas in Report *A*] in the proposed Manual would be a grave mistake." After referring to the analysis whereby pile-driving formulas are derived, he declares that "* * * the field of assumptions is so broad as to plague it with the problem to eternity." As a field for assumptions, he mentions types, shapes, and sizes of piles; materials of which piles are made; and hammers, methods of installation, types of equipment, job and weather conditions, condition of equipment, types of soil and combinations thereof into which piles are driven, types of structures to be supported, etc. He has made clear the difficulties of devising any formula or formulas, and points out that

> "* * * the science of soil mechanics requires complete subsurface data and samples of soil for identification and laboratory tests. This is a common-sense approach to every foundation problem except that the laboratory test is superfluous in such a high percentage of cases that it is necessary only to acknowledge its necessity in special problems."

After comment on the futility of depending on pile formulas, Mr. Mohr states that

> "Since engineers and others will use a pile formula of some kind, it seems that the 'Engineering News' formula denominator $(s + 0.1)$ should be inserted in Report *B* and its limitations stated. It has been established, in the United States, at least, by years of common usage, with no record of inadequacy against it, when properly used."

Mr. Cummings contributes a masterly discussion[101] of the entire subject. He lists and discusses five types of pile formulas, and points out the weaknesses and shortcomings in each case. In his opinion "* * * the publication of Report *A* in a Manual of Engineering Practice would be a serious mistake." It is his opinion that the body of the Manual should contain a series of plain state-

[98] *Proceedings*, Am. Soc. C. E., December, 1941, p. 1954.
[99] *Ibid.*, January, 1942, p. 169.
[100] *Ibid.*, p. 170.
[101] *Ibid.*, p. 172.

ments on the subject of pile formulas, similar to Report *B*, and that mathematical derivations should be in an appendix, rather than in the body of the Report; and further, that the mathematics should cover static as well as dynamic formulas. Above all, he urges that the Manual should not present one formula as being thoroughly reliable and all others as being entirely unreliable.

Professor Terzaghi refers[102] to "the penetrating and conclusive mathematical analysis of the subject by A. E. Cummings,[68] M. Am. Soc. C. E." He points out deficiencies in Report *A* and states that "Report *B* deals with its subject strictly in accordance with the scope and purpose of the Manual." However, in his opinion, "* * * it would be advisable to make some of the statements more specific and to eliminate whatever may encourage the reader to use theories and procedures that have not yet stood adequate tests of practical application." Professor Terzaghi contributes an able discussion of the problem, including the shortcomings of the various formulas, and he points out that some of the "complicated" formulas give results no better than those from the simple "Engineering News" formula. His recommendations, which are covered in a summary, will be a valuable guide to the further work of the Committee.

Mr. Peck[103] states that "On the basis of the data in Table 2,[72] it can be demonstrated by a purely statistical approach that the chances of guessing the bearing capacity of a pile are better than of computing it by a pile-driving formula." He points out that the statistical study which he submits indicates that the use of a pile-driving formula is merely a somewhat inferior method of permitting the laws of chance to operate in the determination of pile capacity. His point of view is worthy of serious consideration.

Professor Casagrande,[104] who has so ably contributed much to the work of the Committee, declares that "The question of how to treat the chapter on pile formulas is indeed a difficult one, particularly in view of the desired standard expressed in the first paragraph of the Manual manuscript" (to enunciate sound principles which are based on established facts, and to avoid stating rules or giving formulas that might lead to its unintelligent use). He warns that rigorous adherence would eliminate all pile formulas, since they are not based on "established facts."

In his discussion he applies the time-honored analogy of the impact of billiard balls, as showing that the theory of restitution propounded by Newton does not apply to the case of a hammer striking a pile, confirming Mr. Cummings' remarks in this matter. Professor Casagrande shows three typical curves of penetration and elastic rebound of a steel pile, with the same energy per blow observed in all three cases. The penetration curves are, however, quite different. He concurs in the opinion that the "Engineering News" formula may be preferable to the more complicated formulas, particularly those containing the elastic compression of the pile. His opinion is partly due to the fact that

[102] *Proceedings*, Am. Soc. C. E., February, 1942, p. 311.

[68] "Dynamic Pile Driving Formulas," by A. E. Cummings, *Journal*, Boston Soc. of Civ. Engrs., January, 1940, Vol. XXVII, No. 1, p. 6.

[103] *Proceedings*, Am. Soc. C. E., February, 1942, p. 323.

[72] *Ibid.*, September, 1941, p. 1393.

[104] *Ibid.*, February, 1942, p. 324.

complicated formulas tend to inspire more (unwarranted) confidence. He views the problem from a very practical standpoint when he states that pile loading tests cannot be made a general requirement, although on large projects they should be used and would pay for themselves, whereas they are often too expensive for small projects.

Professor Casagrande outlines his own approach to the foundation problem, which may be briefed as follows:

(*a*) Thorough subsoil exploration (Mohr and Hvorslev);
(*b*) Digest of the information to determine foundation type;
(*c*) In case a pile foundation is chosen, select the type of pile and allowable load;
(*d*) Allowable load frequently fixed by empirical rules in "building codes";
(*e*) In absence of load tests, pile formulas are supposed to determine final resistance—no provisions in these formulas against overdriving; and
(*f*) Empirical rule:

> "A pile driven to the maximum permissible resistance that will not harm the pile can be loaded safely to the maximum allowable loads permitted in building codes."

This is a common-sense approach to a problem, and must necessarily be used with discretion.

Professor Dunham[105] has stated the essence of the problem that is before the Committee. He states that

> "* * * the Committee is faced with the question of what to advise engineers to do when designing and building small structures on compressible soils for which suitable tests are not, or cannot be, made in advance. * * * the engineers concerned with the job will almost surely decide upon a safe load for the piles, and then they will select a formula of their own choosing to guide them, striving to realize the hoped-for bearing capacity, meanwhile feeling that the Committee has left them adrift."

He advises that the Committee state in the strongest possible language that

> "* * * denied the assistance of proper tests in advance, the designer must do the following:
>
> "(1) Accept no alternative to a course of conservatism because he must deal with factors that affect the safety of life and property; and
>
> "(2) Let the hidden costs due to this conservatism be a financial burden carried unconsciously by the owner because of the lack of adequate information."

Conclusion.—In the twelve months succeeding the publication of the Report, discussion has appeared in seven issues of *Proceedings* from September, 1941, through March, 1942. If it were to be compiled, the complete work would comprise a book of about 140 pages. Being in fact a progress report (as distinguished from a final report) this material is not to be published in *Trans-*

[105] *Proceedings*, Am. Soc. C. E., March, 1942, p. 445.

actions, and therefore not to be collated except in so far as parts of it may be incorporated in the Manual. Those who are interested, therefore, should take special note of saving their copies of *Proceedings* containing this material.

In the light of what the discussion has revealed, and with much other information at hand, the Committee must now endeavor to rewrite Chapter 9 of the proposed Manual, which was submitted in alternative forms in its Progress Report. It seems advisable that the subject be approached from the standpoint of the capacity of the soil to carry load from an individual pile, and of determining when any given pile has been driven sufficiently to develop the capacity of the soil to carry the desired load. Pile formulas are only a part, and should not displace the larger problem. The Committee, the Society, and the engineering profession owe much to those engineers who have given so liberally in these discussions of their time and knowledge, based on experience, in attempting to solve this problem.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

TIMBER FRICTION PILE FOUNDATIONS

Discussion

BY RAYMOND D. MINDLIN, ASSOC. M. AM. SOC. C. E.

RAYMOND D. MINDLIN,[9] ASSOC. M. AM. SOC. C. E.[9a]—An unfortunate error made by the author invalidates most of the conclusions drawn from the mathematical portion of his investigation. A solution of the equations of elasticity, known as the Boussinesq solution, is used in this paper, in an attempt to establish a rational estimate of the stress distribution in the soil around a friction pile. The applicability of the classical theory of elasticity to stresses in soils is debatable, but may be allowed as a first approximation on the basis that no better theory exists. However, once this barrier is passed and the investigation enters the realm of elasticity theory, it is not permissible to violate the mathematical and physical laws which immediately become effective.

It has been observed before[10] that, when the Boussinesq solution is integrated along a line parallel to the direction of the applied force, as the author has done, additional distributed loading is automatically introduced on the surface. The resulting stresses in the interior of the body are then those due not only to the line load penetrating into the body, but also to the distributed surface load.

The author implies that his formulas (Eq. 2c, for example) represent stresses at depth Z due to the distribution of delivered load P' (Eq. 2a) extending from the butt to the tip of a pile of length L. He should state, at this point, that the form of P' is either an assumption, or is based on experimental or theoretical results which he should supply. (Actually, the determination of P' is the most difficult part of the problem.) Furthermore, Eq. 2c gives, in fact, the distribution of s_v at (and only at), a depth Z due to (1) that part of P' (and only that part) between the surface and depth Z, plus (2) a surface distribution of s_v and q_v.

NOTE.—This paper by Frank M. Masters, M. Am. Soc. C. E., was published in November, 1941, *Proceedings*. Discussion on this paper was published in *Proceedings*, as follows: February, 1942, by E. H. Connor, M. Am. Soc. C. E.; and April, 1942, by Messrs. Glenn B. Woodruff, Jacob Feld, G. G. Greulich, and G. S. Paxson.

[9] Asst. Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[9a] Received by the Secretary March 3, 1942.

[10] "On the Distribution of Stress Around a Pile," by R. D. Mindlin, discussion of papers, Section E—Stress Distribution in Soils, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., Vol. III, 1937.

The fact that the surface distribution exists in the author's solution is masked by the fact that, in Eq. 2c, the symbol Z represents both the coordinate of the plane on which s_v is to be calculated and the depth to which the line of load extends. If Z is allowed to approach zero in Eqs. 2c and 3c, s_v and q_v also approach zero; but this is not because no surface loading s_v (or q_v) is implied by the author's integration. Because of the dual meaning of Z, one cannot move the plane on which the stress is to be examined without changing the length of the line of load. Thus, if it is desired to use Eq. 2c or Eq. 3c to investigate the stress at the surface, one will, at the same time, automatically reduce the length of the line of load to zero (that is, the entire solid is unloaded), resulting in no stresses in the solid at all, and, of course, no stresses at the surface. The expression for s_v the author would have found, if he had prosecuted his incorrect theory rigorously, is (confining attention to the case $R_c = 1$, for simplicity):

$$s_v = \frac{P\,r}{2\,\pi\,L}\left\{\frac{Z^3}{r^2\,[r^2 + Z^2]^{1.5}} - \frac{(Z - Z')^3}{r^2\,[r^2 + (Z - Z')^2]^{1.5}}\right. \qquad (5)$$

in which Z is the distance below the surface of any point, and Z' is the distance, below the surface, to which the line of load extends. It is then seen that if no distinction is made between Z and Z', the author's result is obtained. In Eq. 5, however, it is seen that the stress does not vanish at the surface ($Z = 0$). Hence the stress distribution found by applying the Boussinesq solution is one due to a line distribution of load, plus a surface distribution which does not actually exist. The author's formulas represent only a part of the results to be expected from integration of the Boussinesq solution, and, since the Boussinesq solution does not apply to his case, even the part that he gives is incorrect.

Furthermore, in Eq. 5, Z' should be taken equal to L (the length of the pile); otherwise, if it is taken equal to Z, it is assumed incorrectly (and the author has implicitly done this) that the portion of the pile extending below the plane on which the stress is calculated does not affect the stress on that plane.

The author refers to a solution of the elasticity equations by the writer[2] and states that it does not check the test results quoted in the author's paper. This is not surprising on several counts, but chiefly because the solution and the tests do not deal with the same problem. The writer's solution is for the stresses due to a force applied at a point in the interior of a semi-infinite elastic solid, just as the Boussinesq solution is for a force at a point on the surface of the semi-infinite solid. Neither solution deals directly with stresses due to line distributions of forces, although both may be extended to such cases, by integration, with the important reservation that the Boussinesq solution is restricted to loads distributed on the surface of the solid.

The author refers to a thesis by J. Ruderman as mentioned in a paper by D. M. Burmister.[3] Mr. Ruderman's work contains the correct integration

[2] "Force at a Point in the Interior of a Semi-Infinite Solid," by Raymond D. Mindlin, *Physics*, May, 1936.

[3] "Proceedings of the Purdue Conference on Soil Mechanics and Its Applications," Purdue Univ., Lafayette, Ind., September 2 to 6, 1940, p. 339 (citing also "Stress Distribution Around a Loaded Pile," by J. Ruderman, Thesis No. 500 submitted to the Dept. of Civ. Eng., Columbia Univ., New York, N. Y., in 1939, in partial fulfilment of the requirements for the degree of Master of Science).

for a uniform line distribution of load in the interior of the semi-infinite elastic solid. This thesis gives the results the author should have obtained for the case $R_c = 1$. Table 10(*a*) of the paper contains only a small portion of Mr. Ruderman's results and it is then stated that "From Table 10(*a*) * * * it has been possible to compute a curve of vertical shears around a pile at midheight. In Fig. 11 this curve is shown * * *." On the contrary, it is not possible to perform such a computation from the data given in Table 10(*a*), with the result that the curve in Fig. 11 designated as "Mindlin Distribution for Uniform Load per Linear Foot" is incorrect.

The author uses the principle of superposition to find the distribution of stress for a group of piles. It is important to emphasize that the distribution, along a pile, of load transfer from pile to soil, is influenced by the presence of neighboring piles, so that superposition is not strictly permissible in this case.

The real problem, for either a single pile or a group of piles, is to find the distribution, along the pile, of the load transfer from pile to soil. The remainder is comparatively simple, as long as elasticity is assumed. A study such as Mr. Ruderman's, even though an assumption as to this distribution is made at the start, is useful because information can be obtained from it that may be helpful in a more comprehensive attack on the problem. Mr. Masters, however, has produced a set of formulas whose only connection with the problem he has assigned himself is that they accidentally conform in part with his experimental data.

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DISCUSSIONS

THE GREASE PROBLEM IN SEWAGE TREATMENT

Discussion

BY HARVEY F. LUDWIG, JUN. AM. SOC. C. E.

HARVEY F. LUDWIG,[7] JUN. AM. SOC. C. E.[7a]—A comprehensive review of the grease problem in sewage treatment is presented in this paper. The authors have necessarily limited their discussion of new analytical procedures for the determination of grease; but they do emphasize that there is considerable current interest in the possibility of revising and improving the old standard methods procedure. The writer recognizes that detailed discussions of the various analytical operations making up the proposed new procedures are of interest chiefly to chemists. However, a brief consideration of the fundamental principles involved in these methods should be of considerable interest to the engineer. Such consideration promotes an understanding of the basic characteristic properties and nature of greasy substances, an understanding obviously prerequisite to an intelligent solution of an engineering grease-disposal problem.

Considerable confusion exists in the literature dealing with the analytical determination of grease, and perhaps most of this arises from the lack of an exact definition of just what the term "grease" should include. Some have questioned, for example, that both organic derivatives (lard, butter, soaps, etc.) and mineral derivatives (kerosene, fuel oils, etc.) should be included; others suggest differentiation between "greases" and "oils," according to whether they are solid or liquid at room temperature. Again, a purely theoretical definition of grease on the basis of chemical constituents leads to confusion because certain substances included by such definition have little or no grease-like qualities. Further, such a variety of substances of greatly varying properties have some grease-like qualities that it is obviously impossible to devise a single test that will recover them all.

What interests the sanitary engineer are those substances which are likely to occur in appreciable quantities in domestic or industrial wastes and which

NOTE.—This paper by Almon L. Fales and Samuel A. Greeley, Members, Am. Soc. C. E., was published in February, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Arthur D. Weston, M. Am. Soc. C. E.

[7] Graduate Student in San. Eng., Univ. of California, Berkeley, Calif.

[7a] Received by the Secretary March 27, 1942.

will be of sanitary significance—that is, those substances the particular properties of which are important in sewage treatment and disposal practice. "Substances of sanitary significance" again include a multitude of substances of multi-varied properties. These substances generally have one property in common, however, in that they tend to be removed from suspension and to agglomerate, usually at the surface of the medium (hence their "sanitary significance") in which they were originally dispersed in colloidal state. The writer is of the opinion that this property should form the basis of the accepted method of grease determination; and, further, that having adopted such a method, grease should be defined thereafter as just those substances which are recovered by the method.

The old standard methods (1)[7b] procedure for determination of grease consists essentially of strongly acidifying the sample with hydrochloric acid, evaporating the sample to dryness, and then extracting the grease from the dry residue with some solvent such as chloroform or petroleum ether. A Soxhlet apparatus is generally used for making the extraction. The procedure is based on the following assumptions: (1) The operation of acidification converts any non-solvent-soluble greases (calcium and magnesium soaps) to solvent-soluble greases (hydrogen soaps or fatty acids); and (2) the operation of evaporating the sample to dryness (which conveniently removes the water, leaving a dry residue on which the extraction is easily made) does not appreciably alter the state or nature of the grease materials in the system.

Various investigators (9),[8] particularly Messrs. Pomeroy and Wakeman,[9] recently have made a critical study of the standard methods procedure, and have discovered that numerous errors result from the operation of evaporating to dryness, because changes in state do occur. These changes may be classified, according to Messrs. Pomeroy and Wakeman, as follows: (*a*) Volatilization of grease; (*b*) reversion of fatty acids to insoluble soaps; (*c*) production of ether-soluble matter (from combined fatty acid radicals without greasy characteristics, as egg yolk); and (*d*) treatment of oils by oxidation and polymerization. Messrs. Pomeroy and Wakeman subsequently devised a novel "wet extraction" procedure (similar to that commonly used in analysis of oil-field wastes and of milk) which eliminates the troublesome evaporation operation. The solvent is added directly to the acidified sample, and the entire mass is shaken thoroughly. After the solvent separates from the remainder of the medium (special technique may be necessary to induce this separation), it is withdrawn, filtered, and evaporated, leaving the residue to be weighed. By this procedure the aforementioned errors are largely eliminated. It should be noted, however, that in both the standard methods procedure and in the Pomeroy-Wakeman method, all of the solvent-soluble materials originally present are recovered, regardless of their grease-like qualities or sanitary significance.

While employed as engineering assistant for the East Bay Cities Sewage

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7b Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography of the paper (see Appendix).

8 "A Suggested Procedure for the Determination of Grease," by H. F. Ludwig, *Sewage Works Journal*, Vol. 13, July, 1941, p. 690.

9 "Determination of Grease in Sewage, Sludge, and Industrial Wastes," by Richard Pomeroy and C. M. Wakeman, *Industrial and Engineering Chemistry* (Analytical Ed.), Vol. 13, 1941, p. 795.

Disposal Survey,[10] the writer, under the direction of Charles Gilman Hyde, Harold Farnsworth Gray, and A. M. Rawn, Members, Am. Soc. C. E., developed a procedure[8] based on the tendency of grease materials to be removed from suspension and to collect on the surface. This procedure, again an adaptation (being commonly used in soap chemistry), has been called the "boiling-chilling" procedure by the writer, because these two operations are of basic importance. Essentially, the method comprises acidifying the sample, boiling for a few minutes, chilling in the refrigerator (overnight), filtering, drying the filter paper and retained solids, and extracting the grease from these with a suitable solvent; either directly or by means of a Soxhlet. The operation of boiling serves a dual function: (1) To insure complete conversion of insoluble soaps to soluble fatty acids (it is the experience of soap chemists that, in a cold process, conversion is incomplete even under highly acid conditions); and (2) to concentrate the grease in a surface layer or in surface globules (if desired, a large part of the underlying liquid may be drawn off at this point). The operation of chilling serves to congeal the grease globules so that they may be separated later from the system by the operation of filtration. Any solvent-soluble materials which are present in the liquid drawn off after boiling or which are not retained on the filter paper during filtration thus will not appear in the final grease residue.

Messrs. Okun, Hurwitz, and Mohlman (9) subsequently tested this procedure, and developed an improved technique by utilizing certain inert substances to increase mechanically both the rate of filtration of the chilled liquid and the rate of extraction of grease from the dry retained solids. It is interesting to note that the method of grease determination tentatively approved by the Committee on Research of the Federation of Sewage Works Associations in its recent report[11] is the aforementioned boiling-chilling procedure, as improved by Messrs. Okun, Hurwitz, and Mohlman, the final extraction being made in a Soxhlet apparatus with either petroleum ether or hexane.

The question as to which solvent is best suited for extracting the grease from the dry retained solids is a problem in itself. Here again the question of just what is included in the term "grease" is pertinent. The best solvent would be that which most nearly extracts these substances to the exclusion of others. Messrs. Pomeroy and Wakeman have studied the extracting properties of a number of solvents, and recommend hexane as extracting a minimum of nongrease materials and yet having sufficient solvent properties to extract most of the commonly recognized greases.

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[10] "Report Upon the Collection, Treatment and Disposal of Sewage and Industrial Wastes of the East Bay Cities, California," by Charles Gilman Hyde, Harold Farnsworth Gray, and A. M. Rawn, Berkeley, Calif., June 30, 1941.

[11] *Sewage Works Journal*, Vol. 14, 1942, p. 317.

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DISCUSSIONS

DEVELOPMENT OF TRANSPORTATION IN THE UNITED STATES

Discussion

BY F. R. SCHANCK, M. AM. SOC. C. E.

F. R. SCHANCK,[22] M. AM. SOC. C. E.[22a]—In presenting the subject of transportation in the United States, the author brings before the Society one of those basic enterprises, the discussion of which properly includes the broadest aspects of engineering. The consideration of such matters tends to expand the viewpoint of members beyond the rather narrow confines of purely technical engineering. It helps to remind them that they are primarily citizens of communities and of the Nation rather than simply members of an academic debating club.

As the author indicated, a thorough study of the development of transportation in the United States requires research into history. However, the writer does not believe that merely a reference to the "First Stone" of the Baltimore and Ohio Railroad brings to the subject the full light which history can furnish. Besides history, it is necessary to consider economics, geography, law, and industrial and inventive growth, not to mention finance, and politics. In fact, it appears wise to revive one's recollection of the true function of engineering. The definition runs something like this: The study and the utilization of the forces and the materials of nature for the benefit of humanity. The last five words furnish the reason for the very existence of engineers. The author does not seem to have emphasized that ideal in his brief for railroads.

One very interesting and important type of transportation in the United States is not covered by the author; that is, city "mass transportation." During the life time of many, city street cars were drawn by animals—horses, and mules; then many lines used cable traction. Soon practically all these were supplanted by electricity and now, except in a few of the largest cities, buses are used for nearly all surface transportation. Until very recently many

NOTE.—This paper by J. E. Teal, M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: February, 1942, by Messrs. Fred Lavis, and William J. Wilgus; March, 1942, by Joseph B. Eastman, Esq.; and April, 1942, by Messrs. J. L. Campbell, W. W. Crosby, and W. B. Irwin.

[22] Cons. Engr., Portland, Ore.

[22a] Received by the Secretary March 30, 1942.

street railway companies were constantly emphasizing the fact that passengers may be carried more cheaply by electric street railways than by any other present method. However, as the president of one of the New York surface car lines has said, when requesting from the Board of Local Transportation permission to substitute buses for electric rail cars, "The people like to ride on rubber."

The fact that mass transportation has actually passed through the initiation, the zenith of prosperous operation, and now the almost complete elimination of a universally used type of plant and equipment within a comparatively short time, gives the opportunity to analyze such a development. An unbiased and critical study by engineers of a system so nearly parallel with steam railways should give accurate and helpful data as to the probable destiny of the latter. One assumption that frequently predicates discussion of transportation development is that the system that costs the least will inevitably dominate. The fact that cost alone may not necessarily satisfy the public is demonstrated by the transitions in mass transportation.

Some years ago, when the term of the franchises for the street-car lines of Portland, Ore., had expired, there was such violent and widespread disagreement expressed by the citizens as to renewing the franchises that the City Commission hesitated to take any action. Finally the mayor appointed a committee of citizens with a request that this body make some suggestion as to whether the old type of franchise be offered the Street Railway Company, whether a radically different set of conditions be demanded, or whether the existing company be considered as having a status no different from any other applicant for a franchise. The writer, being a member of this committee, had access to considerable public sentiment on the matter of mass transportation. Preferences for different methods of solving the mass transportation problem of Portland did not seem to be based primarily on costs to the riders.

At this same time, the writer was chairman of a committee of the City Club of Portland which had been appointed to analyze and digest the various suggestions and programs that were advanced as the solution of the mass transportation problem. This committee spent several months in collecting data from all cities of a size comparable with Portland. The committee's position also furnished the opportunity to secure statistics as to the apparent indication that cost may not be the primary factor in the minds of the public when purchasing transportation.

Several thousand questionnaires were placed in the hands of individuals of many groups which together seemed to give a fair cross section of public opinion. These included all the employees of certain laundries, banks, department stores, service clubs, factories, and office building occupants. Such questions as their ownership of automobiles, as well as their class of work, indicated quite clearly the economic status of the individuals. The majority of every group showed that some other consideration besides the amount of fare would determine their selection of means of transportation, except the employees of an interurban street railway company which has since abandoned passenger service.

As long as the selection of a mode of transportation of either freight or passengers is a matter of personal choice, the evidence just cited seems worthy of consideration by all common carriers. The limitations of the factors described in the Transportation Act creating the Board of Investigation and Research appear too narrow for a wise report on such a basic subject. The social and cultural effects of limited or abundant transportation facilities are of the utmost importance. The restraint or the development which a particular region or community may experience by absence of impartiality in its transportation services or charges as compared to other places and a multitude of complex problems all have roots in transportation.

The author has brought each other type of transportation, whose development he has described, into its relationship to railways. He writes of "normal conditions" in which railroads require large amounts of materials for operations and expansion. This normal condition apparently means when railroads have a substantial monopoly of transportation.

To have an adequate picture of the entire history of rail transportation as it is now known, it is necessary to go back at least 50 years prior to the occasion, noted by the author, of the inauguration of the Baltimore and Ohio Railroad construction. The earlier date covers that time in the British Isles when rails were laid upon which horse-drawn vehicles transported freight and passengers, and especially coal, in some of the mining regions of that country.

Where these "rail ways" were constructed along the routes of public highways they seem to have been given the same status as any other toll road, probably with the idea that those toll payers who had their own vehicles equipped with suitable wheels could have the privilege of transporting freight over the improved roadbed. Some of the rail systems permitted the use of vehicles with ordinary type wheels; others required flanged wheels.

The unique fact that at this same time the newly invented steam engine was adapted for locomotion brought about the revolutionary changes which have resulted in the modern railway systems. When steam locomotives were used as the tractive force, it became obvious that several unrelated users could not satisfactorily operate vehicles over these improved "toll roads." The implications of the origin of railways remain even after the complete evolution of the private property idea which now almost obscures the original status. Both the right of eminent domain which railroads possess and their designation as common carriers retain the germ of the origin of railroads as public highways. Permission to construct rail toll roads and to collect revenue for their use was by grants limited in time. The fee title remained in the Crown or other sovereign. Certain of the earlier charters granted to railway companies in the United States contain language which indicates the origin of this governmental status of the railway system.

In this country especially, the evolution to private ownership has been complete so that for many decades court decisions have indicated clearly that railroads are private property. This status of private property in the hands of railroad owners makes a distinction inevitable in the treatment which the railroads must expect in their dealings with governmental bodies. The author

seems not to have taken this into account in his comparison of the status and problems of railroads with water, air, and highway transportation systems. All of these are free, legally, to be used by any one wishing to do so. On the other hand, no one can use the railway except the owners.

No individual or group receives any income from the use of the highways, improved air routes, or waterways. Apparently there is no comparable basis for computing the public funds to be spent on public improvements or the taxes to be levied on private property or any other of the subjects which the author indicates are matters of concern.

Although it may sometimes appear to be unjust, yet it does seem one of the requirements of nature that "iniquity of the fathers" is visited "upon the children unto the third and fourth generation." Going back much less than three generations when the railroads were substantially in complete control of all transportation, the president of one of the largest systems was most widely known by his exclamation, "the Public be damned." Even as recently as 1900 or 1910, such apparent elementary rights, which the shippers now possess as designating a route for freight shipments, were not at all privileges assured to those who paid the bill. As indicated in the case of the preference of the public as between street cars and buses even at higher cost, the convenience of door-to-door freighting service, deliveries in a reasonable and fixed time after shipment, and more frequent services all helped to build up the trucking business at the expense of railways.

In other words, more acceptable, convenient, or cheaper transportation was provided. Just as faster, although not cheaper, shipments by rail in the early part of the nineteenth century put canal boat operators out of business, so now an improvement over railways has eliminated many branch lines and caused loss of traffic, both freight and passenger, to automobiles, airplanes, and pipe lines.

There does not seem to have been any reason why the railroad companies themselves might not have foreseen the advantages that the autotruck provides for certain types of business and have used these to supplement their rail business. Instead of this, during the first few years of autotrucking, railroads combated in every way possible the additional convenience which the trucks provided the public. After it became apparent that the truck transportation was here to stay, not infrequently railroads have expended considerable sums in payment for franchises or other intangible rights to established truck or bus operators.

The question of inequity of taxation of railroads seems to be no more complex than that by which highways are financed. Although highways are used for all types of transportation, for access, and for the general convenience of the adjacent property owners and the public, they are financed almost solely by a tax on gasoline imposed on one class of users. In fact, the construction of highways has often provided great traffic outlets for railroads. In the same manner, much of the expenditures for harbor improvements, which the author classes as competition unfair to railroads, has built up water traffic most all of which is hauled to ports by railroads. The great bulk of the

foreign commerce of the United States has, in the past, been lumber, wheat, cotton, meat, and food products which involved long rail hauls. Without the improved harbors many industries and the communities they support, with the railroads benefiting, could not have reached their present magnitude.

Soon after the First World War, a Royal Commission on Transport was appointed by the British Government to study all phases of the subject. The reports of this commission are illuminating even if some conditions in Great Britain are very different from those in the United States. The authority and scope of its investigations apparently were much broader than those of the Board of Investigation and Research which the President has been authorized to appoint. The Royal Commission had in its personnel not only those technically qualified to represent various types of transportation but also those whose viewpoint was more that of the well-informed public. Of course, this later viewpoint is the one which should predominate in dealing with matters as vital to all people as transportation.

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DISCUSSIONS

HYDRAULIC DESIGN OF DROP STRUCTURES FOR GULLY CONTROL

Discussion

BY MESSRS. L. STANDISH HALL, AND J. E. CHRISTIANSEN

L. STANDISH HALL,[12] M. AM. SOC. C. E.[12a]—The research work reported by the authors to determine the proper design of drop structures is noteworthy both for the way in which the experiments were made and for the manner in which the results have been presented. The existing knowledge of the proper length of the basin below the drop has been extended so that a proper design is now possible under a wide range of conditions.

One of the most important features, in the writer's opinion, was the development of the longitudinal sill, which has proved very effective in eliminating bank scour below drops. It is remarkable that so simple a device can be so effective in solving a troublesome problem. Twenty drop structures built under the writer's direction have operated very satisfactorily and show no signs of bank erosion. Some of these structures, under flood conditions, are described elsewhere.[13]

The authors have suggested several names descriptive of these structures and have adopted the term "stilling basin." The terms "roller basin" or "tumble bay" would be more descriptive of the action of the water in passing through the basin. The action greatly resembles that occurring in a jump basin at the toe of a dam where the basin is too shallow.[14]

Mention has been made of the difficulty of determining equilibrium slopes and stable grades between drop structures. Some data have been gathered on this phase of the problem that may be of interest. At the end of the 1941 runoff season, the silt grade was determined above several of the erosion-control dams on the San Pablo Creek (California) watershed, where an appreciable deposit had occurred. The ponds are only partly full at the present time

NOTE.—This paper by B. T. Morris, Jun. Am. Soc. C. E., and D. C. Johnson, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: April, 1942, by John Hedberg, Assoc. M. Am. Soc. C. E.

[12] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

[12a] Received by the Secretary March 24, 1942.

[13] "Drop Structures for Erosion Control," by L. Standish Hall, *Civil Engineering*, May, 1942, p. 247.

[14] "Study of Stilling-Basin Design," by C. Maxwell Stanley, *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 490.

(March, 1942), and the grade which the silt deposits finally will take after the ponds are eventually filled with detritus is of importance in determining the effective capacity of the structures. Sufficient observations have not been made to arrive at any definite conclusions, but it appears that the grade taken by the silt deposits is inversely proportional to the size of the drainage area. The maximum size of material carried by the creek also has an effect on the silt grade, coarser material taking a steeper slope. The size of rock fragments is governed in part by the type of rock formation in a specific drainage area. On the small drainage areas ranging up to 600 acres, the silt grade was found to vary between 1.3% and 2.3%, whereas on the larger areas it ranged from 0.25% to 1%. The results from the several areas are given in Table 1.

TABLE 1.—SILT GRADIENTS ABOVE DROP STRUCTURES

Structure No.; SP:	Drainage area (acres)	Estimated runoff[a]	Silt grade (%)	*G* (in.)[b]
(*a*) WEST SIDE, SAN PABLO RESERVOIR				
26	321	253	2.1	12
26X1	321	253	1.8	3
11	177	161	2.3	3
11A	177	161	1.8	3
10	133	131	2.0	8
10A	133	131	2.0	8
(*b*) EAST SIDE, SAN PABLO RESERVOIR				
18	548	408	2.2	1
16	47	63	1.6	1
24	97	110	1.3	1
(*c*) SCOW CANYON				
13	1,210	618	1.0	1.25[c]
(*d*) SAN PABLO CREEK				
201D	8,700	3,000	0.4	6
501N	1,100	640	0.25	...[d]

[a] Estimated 25-yr storm runoff, in cubic feet per second. [b] Material graded from *G* inches down. [c] Material graded from fine silt to a 1.25-in. maximum. [d] Private dam above crossroads; material would pass a No. 10 screen.

After the pond above a drop structure has filled with debris so that sand and gravel pass over the crest during flood discharges, there appears to be a pronounced tendency to scour the central part of the stream bed between the longitudinal sills. On one or two structures, under the writer's observation, the placing of riprap or paving in this location is indicated as desirable within a few years.

This bed erosion has occurred on channels having a steep profile. In many cases the natural longitudinal profile of the stream ranges from 1 to 5%. Calculation indicates that the velocity in the channel below the structure is slightly above the critical. Under these circumstances, the "ground roller" would be greatly reduced in size from that observed in the laboratory model, in which the exit channel was on a relatively flat grade. This problem of bed erosion below the drop would arise on structures for gully control, and would not be of importance generally where the structure was used as a drop in an irrigation canal due to the controlled rates of flow, flatter slopes, and lower velocities in the latter.

J. E. CHRISTIANSEN,[15] ASSOC. M. AM. SOC. C. E.[15a]—Most hydraulic laboratory studies are confined to large important works, where possible savings in

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[15a] Received by the Secretary March 30, 1942.

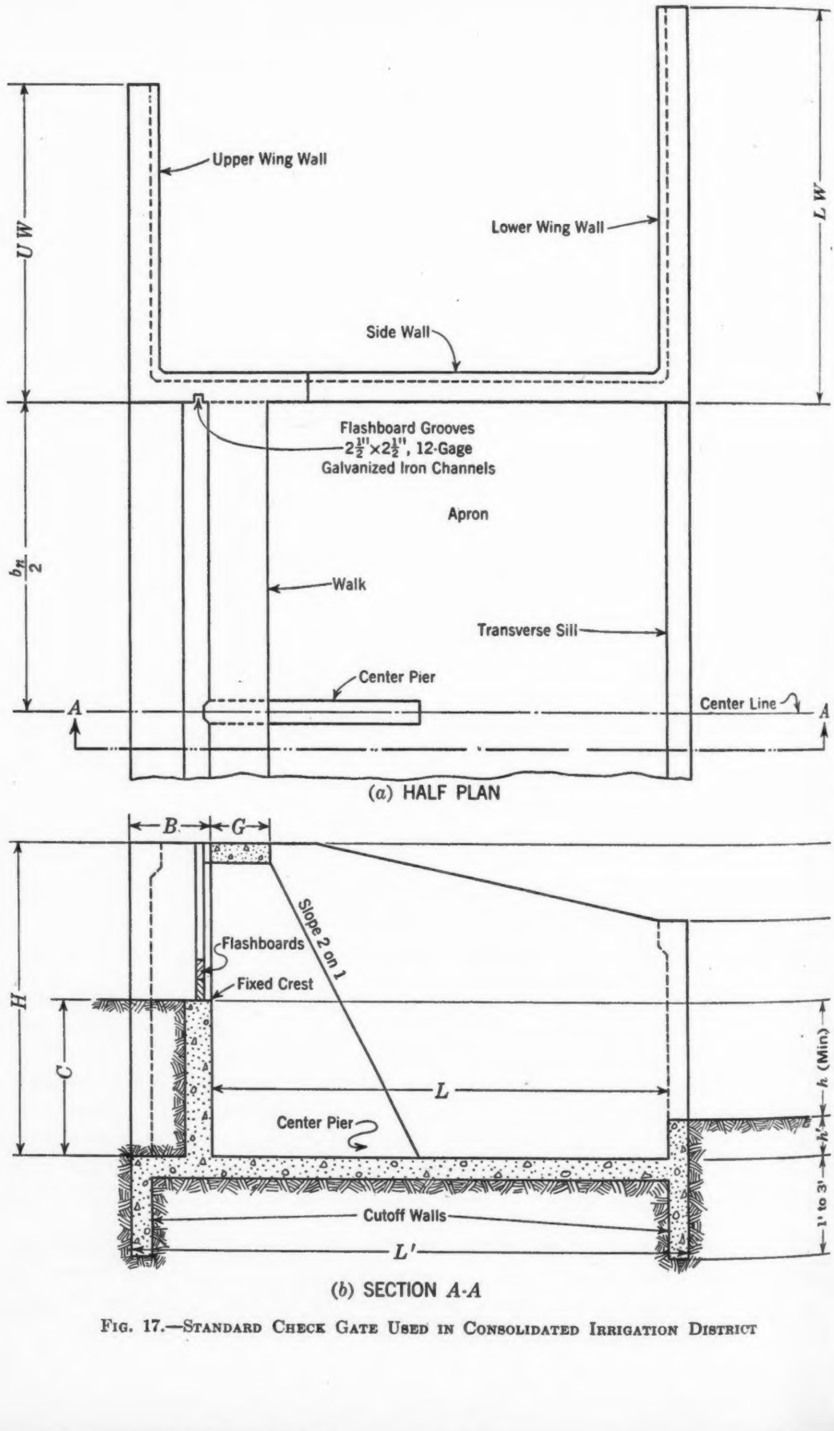

FIG. 17.—STANDARD CHECK GATE USED IN CONSOLIDATED IRRIGATION DISTRICT

a single structure would pay the costs involved. The fact that appreciable savings may be effected by improving the design of small, relatively simple drop structures, used in large numbers, often has been overlooked. The paper by Messrs. Morris and Johnson is apparently the first record of a laboratory study to determine the proper proportions of such structures.

For several years the writer has been interested in the improvement of irrigation systems, especially farm systems and others composed of relatively small canals that ordinarily receive little engineering consideration. Especially significant in this field is the large number of small structures. Since they are individually inexpensive, they are seldom designed carefully. Often they are not designed at all—just built according to the ideas of the construction foreman. That such structures could be greatly improved is clearly evident from field inspections. One might make many improvements by applying sound engineering practice, but still more improvements might result from laboratory investigations of the type described by the authors.

Among the most common structures in irrigation systems in California are combination check gates and drops, which are used to take up the excess grade in the canals, and also to control the water-surface elevation for diversions. In design these resemble the drop structures described by the authors, but they are generally lower and therefore may have higher ratios of d_c/h. They differ also in that they have adjustable crests controlled by flashboards.

Typical check gates are those used by the Consolidated Irrigation District near Fresno, Calif. These structures are representative of those used in districts that have had the benefit of a good engineering department. Originally designed about 1925 by I. H. Teilman, M. Am. Soc. C. E., chief engineer of the District, they have been only slightly modified since then, and have proved entirely satisfactory.

Fig. 17 shows the general design of these check gates; Table 2 gives their principal dimensions. The letter symbols are the same as in Fig. 3. Representative views appear in Figs. 18 and 19. The former is a relatively small structure with single opening. Although it has been in use for some time, there is no visible erosion on the downstream side. Fig. 19 shows one of the larger structures, the total height above the apron being 16 ft. There are eight openings, with a total width between the side-walls of 64 ft, 8 in. The photograph was taken shortly after the structure was completed and before water had passed through.

TABLE 2.—PRINCIPAL DIMENSIONS OF CHECK GATES USED BY CONSOLIDATED IRRIGATION DISTRICT (SEE FIG. 17)

H	h[a]	L'	L	h'	A	B	C[a]	G	UW	LW
3	0	6	4.0	0.5	0	1.50	0	1.0	4	4
4	0.5	7	5.0	0.5	0	1.50	1.0	1.0	4	4
5	1.0	8	6.0	0.5	0.5	1.50	1.5	1.0	5	6
6	1.5	10	8.0	0.5	1.0	1.50	2.0	1.5	6	8
7	2.0	12	9.5	1.0	1.5	2.00	3.0	1.5	7	9
8	3.0	14	11.5	1.0	2.0	2.00	4.0	1.5	8	10
9	4.0	16	12.75	1.0	2.5	2.75	5.0	1.5	8	12
10	4.0	16	12.75	1.0	3.0	2.75	5.0	1.5	9	12
12	5.0	18	13.75	1.0	3.0	3.75	7.0	3.0	9	14
14	6.0	20	15.75	1.0	4.0	3.75	8.0	3.0	9	15
16	7.0	22	17.75	2.0	4.0	3.75	9.0	3.0	9	16

[a] Minimum values.

The main differences between these and the drop structures described by the authors are as follows:

(1) Upper wing walls at right angle to center line of structure;
(2) Crest a short distance downstream from upper cutoff and wing walls;
(3) Adjustable flashboard crest;
(4) Walkway over crest to handle flashboards; and
(5) No longitudinal sills.

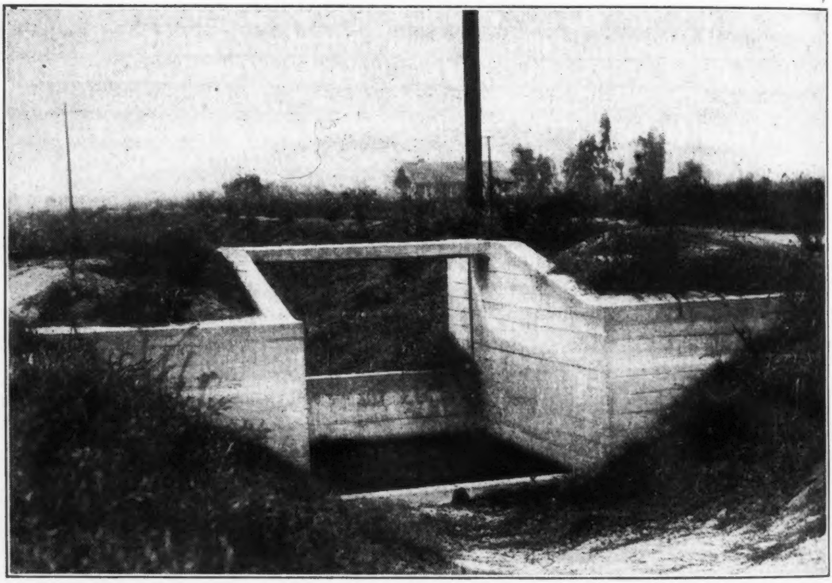

FIG. 18.—TYPICAL CHECK GATE AS USED IN SMALL CANAL—HEIGHT, 7 FT; WIDTH, 7 FT 8 IN.

Originally, the transverse sill was placed from 1 to 2 ft upstream from the lower cutoff wall, but more recently was moved to the end of the structure, thereby increasing the length of the apron. These check gates have been built in heights (top of walk to top of apron) from 3 ft to 20 ft, and in widths (inside of side-walls) from 3 ft, 8 in., to more than 60 ft. Standard practice is to make the widths 4 in. less than an even number of feet so that flashboards can be cut from even-dimensioned lumber without waste. For widths between 7 ft 8 in. and 15 ft 8 in. a center supporting pier is used, and the flashboards span both openings. For greater widths, two or more intermediate piers are used, and alternate ones are provided with flashboard grooves. Normally, wide structures are built with an odd number of intermediate piers, so that all flashboards span two openings.

To minimize the cost of these structures, the practice in the Consolidated District is to standardize the principal dimensions and use collapsible forms made up in panels that can be bolted together. In height, these standard sizes

range from 3 to 16 ft. The length and most of the other dimensions are functions of the total height above the apron. Under special conditions, certain other dimensions are varied.

Longitudinal sills are not used and would possibly offer no advantage, since lateral contraction is eliminated largely by placing the crest a short distance downstream from the upper wing walls. There is no provision for ventilating the nappe. Where the drop is not great, the crest is often submerged, especially at high flows. Under these conditions wave action downstream is severe. The downstream wing walls are made longer than the upstream wing walls to give maximum protection against bank erosion downstream. Additional bank protection for a short distance is sometimes required.

To compare the apron length and the height of the transverse sill with the authors' recommendations as stated in Eqs. 6*a* and 7*c*, the critical depths, d_c, were computed from the design capacities furnished by Mr. Teilman. This allows a freeboard of 1 ft or more under maximum flow conditions. During normal operation there will be one or more 6-in. flashboards in place. An additional comparison, therefore, is made on the assumption that the height of

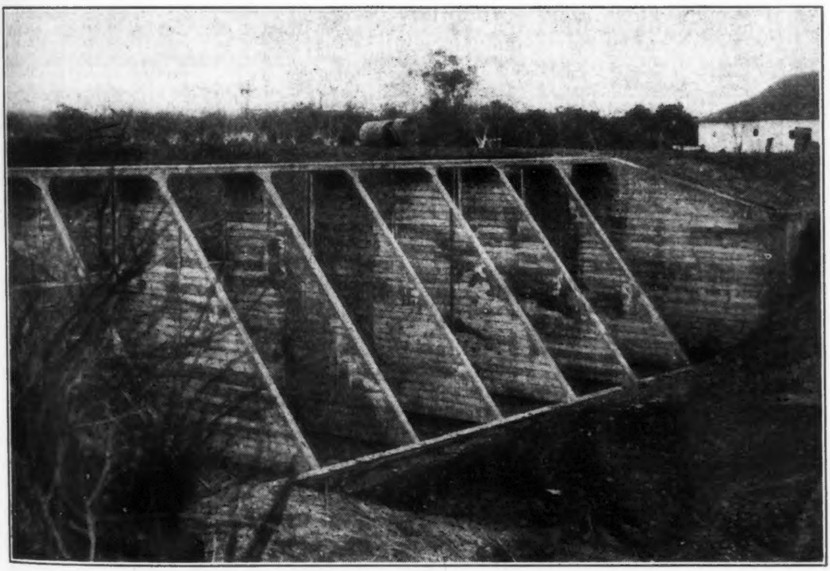

FIG. 19.—LARGE CHECK GATE SHORTLY AFTER COMPLETION—HEIGHT, 16 FT; WIDTH, 64 FT 8 IN.

fall, h, is 1 to 2 ft more than indicated in Fig. 17. Table 3 gives the results of the computations for these comparisons. The 3-ft structure was omitted for obvious reasons. The 3-ft, 4-ft, and possibly the 5-ft structures can scarcely be classed as drop structures. Although Eqs. 5 and 6*a* were developed to give additional length for conditions of low fall and deep flow, obviously these equations should be restricted to ratios of h/d_c that do not exceed 1.0, as mentioned by the authors in their summary of design rules. A relation of the same form

as Eq. 4 might be applicable over a wider range of values of d_c/h. One might use, for example, one of the equations

$$C_L = 2.5 + \frac{2\,d_c}{h} \quad \text{(16a)}$$

or

$$C_L = 2.2 + \frac{2\,d_c}{h} \quad \text{(16b)}$$

Eq. 16*a* gives only slightly higher values of C_L for values of d_c/h between 0 and 1.1, and might apply reasonably well for values of d_c/h up to 2 or more. The use of this equation would give a length of 4.6 and 4.7 ft, respectively, for the

TABLE 3.—COMPARISON OF APRON LENGTH AND HEIGHT OF TRANSVERSE SILL WITH AUTHORS' RECOMMENDATIONS

Actual Dimensions			First Comparison[a]					Second Comparison[b]		
H	L	h'	d_c (max.)	h (min.)	C_L (Eq. 5)	L (Eq. 6a)	h' (Eq. 7c)	h	C_L (Eq. 5)	L (Eq. 6a)
4	5.0	0.5	1.0	0.5	10.30	7.3	0.50	1.5	3.45	4.3
5	6.0	0.5	1.2	1.0	5.03	5.3	0.60	2.0	3.31	5.1
6	8.0	0.5	1.4	1.5	4.08	5.9	0.70	2.5	3.25	6.1
7	9.5	1.0	1.6	2.0	3.74	6.7	0.80	3.0	3.18	7.0
8	11.5	1.0	1.8	3.0	3.31	7.7	0.90	4.0	3.06	8.2
9	12.75	1.0	2.0	4.0	3.14	8.9	1.00	5.0	2.98	9.4
10	12.75	1.0	2.0	4.0	3.14	8.9	1.00	5.5	2.93	9.7
12	13.75	1.0	2.1	5.0	3.01	9.7	1.05	6.5	2.88	10.6
14	15.75	1.0	2.1	6.0	2.92	10.4	1.05	8.0	2.80	11.5
16	17.75	2.0	2.3	7.0	2.89	11.6	1.15	9.0	2.79	12.7

[a] Based on values of d_c computed from design flows, and minimum values of h (no flashboards). [b] Based on same d_c and higher values of h (assuming flashboard in place).

assumed values of d_c and h for the 4-ft structure as compared with 7.3 and 4.3 ft (Table 3). Eq. 16*b* gives almost the same values of C_L as Eq. 5 for values of d_c/h between 0.4 and 0.9, and lower values of C_L outside of this range.

Although the authors recommend shorter aprons, experience indicates that the longer aprons are better than the shorter ones formerly used. Quite likely the depth of water downstream is greater, and the velocities lower, than for the tests. Would this depth influence the desirable length of the apron?

Although any extension of the total length of the structure would increase its cost, the placement of the upstream cutoff wall a short distance upstream from the crest gives greater stability and additional insurance against undermining. This would seem to be a particular advantage in gully-control structures in the West, where during the long periods of drought the soil dries out and shrinks away from the walls. A sudden flow in the dry channel may result in water passing beneath the structure before the soil can become wetted and swell tight again.

The writer wishes to thank Mr. Teilman for furnishing the data used in making these comparisons.

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DISCUSSIONS

EFFECT OF VARIATION OF ELASTIC CHARACTERISTICS ON STATIC UNKNOWNS

Discussion

BY STEFAN J. FRAENKEL, JUN. AM. SOC. C. E.

STEFAN J. FRAENKEL,[6] JUN. AM. SOC. C. E.[6a]—In the writer's opinion, the method presented by Professor Hrennikoff constitutes a valuable contribution to structural analysis and is the first ever suggested along this particular line. The author is to be commended for having devised this thorough and workable solution of a problem otherwise rather beset with difficulties which often make an exact solution impracticable.

It is important to realize the limitations to which this method is subject, as far as its application to the field of design is concerned. Professor Hrennikoff rightly states that in steel structures the variation of stress quantities is very small. Even assuming that the variation amounts to 5%, the resulting additional stress would, in most structures, be smaller than stresses caused by secondary stresses, poorly maintained pin connections, etc. The main field of application, therefore, would lie in the realm of reinforced-concrete structures. Even there, enough uncertainties enter into a design to make unnecessary an investigation along the lines suggested in the paper, except in the very largest structures. It should be possible, however, to put this method to excellent use for purposes of investigation in the following manner:

During the construction of a reinforced-concrete structure, test cylinders of the material could be taken at regular intervals along the structure, and the values of E of these cylinders then could be determined. The maximum value of i, and, consequently, the maximum possible variation of any stress quantity may then be computed for the particular case. Further, the deviations of E and I from their average values within the intervals defined by the roots of Eq. 15 could be investigated, and their algebraic signs would furnish an indi-

NOTE.—This paper by A. Hrennikoff, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1942, by A. H. Finlay, Assoc. M. Am. Soc. C. E.

[6] Detailer, Pittsburgh-Des Moines Steel Co., Des Moines, Iowa.

[6a] Received by the Secretary March 30, 1942.

cation of how closely the theoretical sensitivity coefficient would be approximated. Although this information undoubtedly would be of interest, it would constitute little more than a "post-mortem" examination, and its results would not be applicable to a future design.

A tabulation of sensitivity coefficients for different structures would be a helpful guide to designers, although more in a qualitative than in a quantitative way. Inasmuch as design is still to a considerable degree a matter of judgment, some indication thus would be available to the designer as to where to exercise particular caution. The recommendation that the working stresses be varied in amounts proportional to the sensitivity of the function in question, although theoretically sound, would seem to the writer to be, in effect, wasteful of materials. In framed indeterminate structures it is impossible in any case to stress all members to the allowable unit stress.

To satisfy his curiosity, the writer applied Professor Hrennikoff's method to an arch rib of uniform cross section and of uniform outline. In presenting the results of this investigation, the writer wishes to state that, unless summation is substituted for integration, it will be necessary, in the case of structures with curved center lines, to solve for the roots of algebraic polynomials of the fourth degree, or even higher, which is a tedious process. A computing machine is almost indispensable, inasmuch as the ordinates of the functions are obtained as differences of large numbers. The writer used the slide rule and had to make extensive long-hand computations in order to gain a tolerable accuracy.

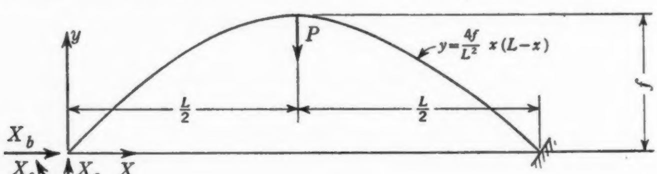

FIG. 9

The arch shown in Fig. 9 was loaded with a concentrated load P at the center, and its equation is

$$y = \frac{4f}{L^2} x (L - x) \qquad (40)$$

The statically indeterminate quantities are shown at the left abutment.

After a routine application of the Maxwell-Mohr method, the values of the unknowns were found to be as follows:

$$X_a = \tfrac{1}{2} P; \qquad X_b = \frac{15}{64} \frac{P L}{f}; \qquad \text{and} \qquad X_c = \frac{P L}{32} \qquad (41)$$

Eqs. 7 become:

$$\Delta_a = \frac{P}{64} \left[\int_0^{L/2} \frac{1}{E I} \left(- 28 x^2 + \frac{60}{L} x^3 + 2 L x \right) dx + \int_{L/2}^{L} \frac{1}{E I} \left(- 92 x^2 + \frac{60}{L} x^3 + 34 L x \right) dx \right] \qquad (42a)$$

$$\Delta_b = \frac{P}{64}\left[\int_0^{L/2} \frac{1}{E\,I}\left(120\,f\,\frac{x^2}{L} - 352\,f\,\frac{x^3}{L^2} + 240\,f\,\frac{x^4}{L^3} - 8\,f\,x\right)dx \right.$$

$$\left. + \int_{L/2}^{L} \frac{1}{E\,I}\left(504\,f\,\frac{x^2}{L} - 608\,f\,\frac{x^3}{L^2} + 240\,f\,\frac{x^4}{L^3} - 136\,f\,x\right)dx\right] \ldots (42b)$$

and

$$\Delta_c = \frac{P}{64}\left[\int_0^{L/2} \frac{1}{E\,I}\left(-\,28\,x + 60\,\frac{x^2}{L} + 2\,L\right)dx \right.$$

$$\left. + \int_{L/2}^{L}\left(-\,92\,x + 60\,\frac{x^2}{L} + 34\,L\right)dx\right] \ldots\ldots\ldots\ldots (42c)$$

Completing the process, the writer found the maximum variations of the indeterminate quantities to be:

$$dX_a\,(\max) = 0.187\,i\,X_a \ldots\ldots\ldots\ldots (43a)$$

$$dX_b\,(\max) = 1.59\,i\,X_b \ldots\ldots\ldots\ldots (43b)$$

and

$$dX_c\,(\max) = 1.92\,i\,X_c \ldots\ldots\ldots\ldots (43c)$$

Generally, in order to obtain the maximum variation, the values of the elastic variables in a parabolic arch with a concentrated center load should be increased from the left abutment to 0.1 L, decreased from 0.1 L to 0.4 L, increased from 0.4 L to 0.6 L, decreased from 0.6 L to 0.9 L, and increased from 0.9 L to the right abutment.

The sensitivity coefficients found are very high for both the horizontal thrust and the moment at the springing, and—to the writer—unexpectedly low for the vertical reaction. Altogether, the parabolic arch appears to be sensitive to elastic variations to a very considerable degree, and a close control of the concrete mixture, in the case of a masonry arch, and of the dimensions, seems advisable.

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DISCUSSIONS

ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY

Discussion

BY MESSRS. N. H. HECK, FRANK NEUMANN, AND JACOB FELD

N. H. HECK,[33] M. AM. SOC. C. E.[33a]—With the development of the program for recording strong earth motions, the Coast and Geodetic Survey has recognized the necessity of full utilization of the results. Its principal contribution has been the determination of velocity and displacement (oscillatory) from the accelerograms so that all the elements of motion are known. The Massachusetts Institute of Technology, Cambridge, Mass., has cooperated in testing the validity of the results and sufficiency of their accuracy for engineering purposes. Little has been done in the application of the results to engineering design. However, the collection of engineering information after a destructive earthquake and the determination of building periods and the periods of other structures, in some cases at intervals during construction, constitute a close relation with practical engineering problems.

For these reasons the paper and previously reported work of Mr. Biot have been especially welcome. Its essential feature is the use of the earthquake spectrum obtained from strong-motion accelerograms with his mechanical analyzer. The spectrum gives a direct measurement of the maximum shear in a building due to a given earthquake. One thing is disconcerting—the high maxima in the relatively weak Ferndale (Calif.) shocks. The motions at El Centro, Calif., due to the Imperial Valley earthquake of March 18, 1940, were considerably greater than had been expected and accordingly one might expect very high maximum values of the spectrum for a major earthquake.

Mr. Biot states that there are a number of reasons for believing that the internal damping of structures is appreciable, and that since he has used an undamped pendulum, high resonance values are due in some measure to lack of damping in the torsion pendulum. The analysis obviously should be extended to include some forms of damped pendulums.

NOTE.—This paper by M. A. Biot, Esq., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1942, by Messrs. George R. Rich, N. J. Hoff, and Merit P. White.

[33] Asst. to the Director, U. S. Coast and Geodetic Survey, Washington, D. C.

[33a] Received by the Secretary March 26, 1942.

In general, it has been held that the tower type of building has special advantages as an earthquake resistant type, but the whip effect on the upper stories cannot be neglected.

The effect of the building on the ground is important from the viewpoint of the Coast and Geodetic Survey strong-motion program. By engineering advice all but about six of the sites adopted were in buildings and the instruments installed in the basements probably are affected by the motion of the building in the earthquake. The fact that the building at El Centro is small has enhanced the value of the conclusions from the El Centro record.

The paper emphasizes the importance of the determinations which have been made of the vibration periods of buildings, since without this knowledge the maximum stress cannot be determined for a given building. The conclusion that change of period damps out resonance is undoubtedly correct, but further investigation of the possibility that prolonged resonance in special cases may result in serious damage is needed.

The paper is stimulating to thought and opens up important lines of investigation. Interest in the subject has not been diverted by the war effort, but few persons are left to work on the problems. Interest in high-acceleration high-frequency vibrations has greatly increased but progress should be made in the entire field of vibrations including the relatively low-frequency long-period vibrations characteristic of destructive earthquakes.

FRANK NEUMANN,[34] ESQ.[34a]—Equivalent acceleration spectrums, as well as those for velocity and displacement, open up a new and vitally important avenue of attack on the engineering aspects of the earthquake problem. The way now seems to be clear to furnish the engineer seismic data in a form that he can put to direct use. The development of spectrums by practical methods has been considered a seismological as well as an engineering problem, and workers in both fields have made important contributions toward current accomplishments.

An interesting feature of the "standardized" spectrum is that, if reduced to "equivalent maximum velocity," the curve will show a constant maximum velocity for all periods greater than 0.2 sec. This would seem to provide an "intensity factor" in standard units for any earthquake motion that has been thus analyzed. This factor, in the case of the Helena north-south component, is 2.4 times the maximum velocity of the ground motion; but the Helena "standardized" curve cannot, as Professor Biot infers,[35] be considered typical of all curves in view of the effect of epicentral distance, for one thing, on ground periods. A notable instance of this distance effect occurred in 1933 when an earthquake originating 183 miles from the San Jose (Calif.) Bank of America Building registered an acceleration, on the thirteenth floor, seven times greater than the recorded ground acceleration of 0.005 g. The approximate resonance period in this case was 1.6 sec. The need for families of spectrums is thus

[34] Chf. of Section of Seismology, Div. of Geomagnetism and Seismology, U. S. Coast and Geodetic Survey, Washington, D. C.

[34a] Received by the Secretary March 26, 1942.

[35] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 163–164.

obvious if earthquake motions are to be adapted to standard patterns. It appears that, for any one earthquake, the acceleration spectrums at different epicentral distances may be restricted in magnitude to the limits of a spectrum obtained very close to the epicenter, barring all geological influences.

It is thought that the apparent multiplicity of periods in the short-period end of the spectrum may be due, in some measure at least, to difficulty in following the acceleration curve during the analyzer work with the necessary precision, especially since resonance is likely to occur between the motion of the torsion-head lever arm (which is manually operated) and the motion of the pendulum. In a similar operation in 1936[36] (to determine the equivalent response of a 5-sec pendulum with a torsion pendulum analyzer) the writer found that it required considerable skill to follow the east-west component of the Helena acceleration curve, moving with the equivalent of the middle speed used in Professor Biot's work, even though the curve was magnified seven times. It is believed that the use of acceleration curves expanded many times, as now used in integration processes in the Coast and Geodetic Survey, would work for greater precision at these critical periods. As to the general form of the spectrum, recent ground period studies[37] indicate that the period pattern changes somewhat with the orientation of the seismograph pendulums, a condition which may be expected to be reflected in the computed spectrums.

With reference to that part of the spectrum below 0.2 sec, the error of Coast and Geodetic Survey accelerometers very seldom exceeds 15% or 20% between 0.1 and 0.2 sec. An experiment is now (1942) under way to test the practicability of shortening accelerometer pendulum periods with a view to increasing the range (downward) for recording true acceleration, and increasing the general recording capacity by decreasing the sensitivity.

It is unfortunate that some of the most valuable acceleration records are difficult to read because of overlapping of curves and occasional overdevelopment of the photographic paper. Overdevelopment is sometimes necessary to bring out the faint records of light spots moving at higher speeds and through larger amplitudes than had been anticipated in the original recorder design. Although these records do not always offer the sharpness of detail and cleanliness which Professor Biot desired in his work, it is nevertheless a fact that only one record to date has been lost to the extent that the data are not available for engineering studies. Available records include two of the Long Beach (Calif.) earthquake of 1933 yielding maximum accelerations of 0.06 g and 0.21 g, and maximum displacements of 20 cm double amplitude; also one of the Imperial Valley earthquake with maximum acceleration of 0.35 g and double amplitude of 40 cm. Comparing these values with the maximum of 0.16 g and displacement of 6 cm involved in Professor Biot's Helena spectrum, some idea may be obtained of the magnitude of the spectrums to come. Moreover, in spite of the magnitude of the recorded motion in the case of the Imperial Valley earthquake, the damage at the recording point,

[36] "A Mechanical Method of Analyzing Accelerograms," by Frank Neumann, *Proceedings*, Am. Geophysical Union, Washington, D. C., May 1 and 2, 1936; also "The Simple Torsion Pendulum as an Accelerogram Analyzer," by Frank Neumann, *Publications du Bureau Central Seismologique International*, Serie A: Travaux Scientifiques, Fascicule 15, 1937.

[37] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C.

El Centro,[38] was "confined to walls that were not reinforced or tied, and to projecting balconies." The amplitude to be expected in an earthquake of catastrophic proportions can only be conjectured.

Concerning the inference that the torsion pendulum is[39] "a less accurate but simpler analyzer" than the proposed electrical apparatus, there will always be a doubt in the writer's mind until actual experience with both types proves the point. Any lack of accuracy in torsion pendulum analyzers would seem to lie in the manner in which the variable acceleration is transferred to the torsion head of the pendulum rather than in the functioning of the pendulum itself; and again, with reference to the statement in Section II of the paper that "it takes an average of eight hours to plot one spectrum curve," engineers are likely to be misled on the magnitude of the task which future workers in this field will face, remembering that even records of intensity VIII earthquakes present a much more formidable problem in processing than one of the Helena type which does not represent an intensity greater than VI. The Helena instrument was mounted on a limestone foundation and not on valley fill where most of the damage occurred in the 1935 earthquake.

Jacob Feld,[40] M. Am. Soc. C. E.[40a]—In characteristic fashion the author prepares a complete and logical derivation of various factors concerning the effect of earthquake vibration on physical structures. The writer is chiefly interested in that section concerning the influence of the foundation on the motion of blocks. In the approach to a solution of that factor, Professor Biot makes the assumption that the soil characteristics do not change during the vibration. Based also upon the assumption that the soil displacements within the limits of the deformations or deflections resulting from the vibrations of the block do not exceed the elastic limit strain, he deduces a formula for the rocking motion of the block, as restrained by the resisting characteristics of the soil.

Practical experience with vibrating structures embedded in soil does not confirm the assumptions made. It is well known in construction practice that, when vibrating machinery such as air compressors are placed on concrete foundations embedded in soil that has been made very heavy to dampen vibrations and thereby reduce their transference to adjacent structures, there is a change in the damping effectiveness with time. The only explanation for the change must be an alteration of the characteristics of the soil. It seems that continuous vibration of this nature increases the density of the surrounding soil. In some instances the writer has noted that a definite gap develops between the faces of the embedded concrete foundation and the adjacent soil. In such instances the vibration transference is considerably reduced, suddenly appearing again when, due to rain or other causes, the gaps are filled in. In one contract the writer designed a support that was kept entirely free from the adjacent soil and was bedded on a layer of cork. A gap was maintained on the four sides of the supporting base over a period of three years, and measurements

[38] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C., p. 3.

[39] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, p. 154.

[40] Cons. Engr., New York, N. Y.

[40a] Received by the Secretary April 8, 1942.

made of the transference of vibration showed a loss of at least 90% of the compressor vibration wave. The period of the compressor vibration was 0.25 sec. Measurements were made by a vibrograph instrument which reported both horizontal and vertical components on a celluloid sheet.

Investigations of the effect of earthquakes and similar vibrations on the lateral pressure of earth have been made by Nagaho Mononobe and Haruo Matsuo.[41] These men found that the pressure of earth against a vertical retaining wall increased quite suddenly to a maximum upon the application of a vibration similar to an earthquake. The amount of increase depended upon the severity of the shock; but in each case the total pressure was found to be maximum immediately upon the application of the vibration, with a slight decrease thereafter. However, the measured pressure after vibration was larger than that before vibration, and the only explanation seemed to be that the material changed in character. This was somewhat substantiated by a series of tests with more compact fills, which did not show as large an increase in pressure from vibration. It also was found that the pressure against rigid walls was increased to a larger extent than pressure against deformable walls. This again seems to indicate that the change in pressure was caused by a change in the characteristics of the fill. Incidentally, the author concludes that the maximum earth pressure under earthquake conditions can be calculated from the usual earth-pressure formula by the use of a density equal to the resultant acceleration obtained by graphic summation of the acceleration of gravity and the maximum seismic acceleration.

[41] "Experimental Investigation of Lateral Earth Pressure During Earthquakes," by Nagaho Mononobe and Haruo Matsuo, *Bulletin*, Earthquake Research Inst., Vol. X, 1932, Pt. 4.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DRAINAGE OF LEVEED AREAS IN MOUNTAINOUS VALLEYS

Discussion

By Messrs. C. O. Clark, W. W. Horner, and Walter T. Wilson

C. O. Clark,[14] Jun. Am. Soc. C. E.[14a]—The drainage of leveed areas in mountainous valleys is soundly analyzed by the author. The approach to the drainage problem by way of rainfall frequency, runoff, and the unit hydrograph has several important advantages over empirical design. First, it is well balanced, in that it is neither grossly inadequate, as when inapplicable empirical relations are applied, nor wastefully over-adequate, as when provision is made for some recent occurrence that is really phenomenal and very unlikely to recur. Second, the solution is approached from the over-adequate side, since cost forces a reduction of adopted size below the first calculation, with the result that, in the usually broad range of acceptable solutions, the more adequate are adopted. Third, the extreme cost of some of the first-calculated solutions leads to an ingenuity of solution not inspired by the approach from the inadequate side. Fourth, utilizing rainfall frequency data, the engineer usually comes to his answer with the deliberate knowledge that the provided capacity may be overtaxed, but that the payment of the damage will be cheaper than its prevention.

The possibilities for utilization of rainfall records coincident with flood stages present a promising opportunity for reduction of cost of pumping installations. The writer, in 1937, made for the U. S. Engineer Department the first studies along this line for the cities of Louisville, Ky., Evansville, Ind., and Cairo, Ill., along the Ohio River, where the procedure is unquestionably sound. The procedure can be utilized for drainage along streams which are so small that crests are reached in as little as 36 hr.

There are many circumstances in which the frequency of rainfalls that have occurred coincidently with flood stage in the main river can be used. Whereas rainfall during damaging flood stages may be too infrequent to study,

Note.—This paper by Gordon R. Williams, Assoc. M. Am. Soc. C. E., was published in January, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1942, by Merrill Bernard, M. Am. Soc. C. E.; and April, 1942, by L. K. Sherman, M. Am. Soc. C. E.

[14] Associate Hydr. Engr., U. S. Engr. Office, Norfolk, Va.

[14a] Received by the Secretary March 30, 1942.

rainfall during moderate and low stages is common enough to indicate the frequency with which it can be expected at high stages.

The writer believes that most of the floodtime rainfall may be treated as a phenomenon independent of the flood-producing rainfall, and he found this to be true in the following example. This depends, of course, upon the difference in time between flood peaks on the main stream and on the tributary. If the time difference is less than 24 to 36 hr, partial dependence is likely, since both floods might result from the same storm. In this case, however, drainage should be estimated for the flood-producing storm, rather than some other synthetic storm, as both together are unlikely.

If the time difference is more than 36 hr, independence of storm causes is essentially attained. Although general conditions are more favorable for rainfall than the average, and rainfall therefore might be more probable than average, the normal movement of storms in the Eastern United States is such that, 2 days after a severe storm, a region is quite likely to be in the high-pressure area, and to be relatively fair, with a reduced probability of rainfall.

However, it is not necessary to assume that the causes are strictly dependent or independent. The degree of dependency can be established. Thus, from the correlation of rainfall and low-river stages, it can be established that rainfall during flood stages is one, two, or several times as probable as average. This conclusion can be extended to higher stages. The probability of heavy rainfall is directly proportional to the total length of experience, almost any rainfall being probable in a long enough period of exposure. If, during a particular period of time, such as floodtime, rainfall were twice as probable as normal, the same rainfall could be expected in that period as in a normal period twice as long as that under consideration.

The duration of main-stream floods is very short (only a few days), so it may take a good many years to build up a year of exposure to floodtime rainfall. If pumping is necessary at the annual flood stage, and the average duration of floods above annual stage is 2 days each, the exposure of an area to floodtime rainfall during 50 years is only 100 days. If rainfall is twice as likely to occur, or two times as probable as normal, the exposure is equivalent to 200 days of normal exposure, and the rainfall to be expected is that which normally occurs once in 200 days.

To establish the relative probability of rainfall during floodtime, it is only necessary to establish a sufficiently low limit of river stage to include the desired number of rainfall occurrences. The probable normal experience for the floodtime period is computed for the same period of time as the aggregate period during which the river was above the critical stage, and is compared with the actual floodtime experience as plotted or computed in the manner suggested by the author. If the computed and observed frequencies do not agree, they will be found to be approximately parallel to each other, when plotted with a logarithmic frequency scale, and one can be expressed as a percentage of the other. Thus, if twice as many occurrences are reported as computed, it is safe to conclude that the probability of rainfall during floodtime is twice the normal. Because of the larger number of hourly amounts recorded, the study of hourly records is easier than the study of daily records. It is not

essential that the rainfall and river stages be observed at the same place. Since the river stage is used only to identify those periods in which floods are likely as compared with the average, in the author's example, recording rainfall records at Scranton and Harrisburg could be utilized with stage records at the locality under consideration—Williamsport.

Data studied for Richmond, Va., on the James River will illustrate the point in question. Rainfall during floodtime at Richmond is rare. Almost 2 days elapse between the rainfall that produces the James River floods and the arrival of the peak at Richmond. By that time, local streams have fallen to low levels, and they rise again only if additional rain falls. By the criterion of the writer, in a preceding paragraph, floodtime rainfall should be independent and normal.

A study of the hourly and daily rainfall records of the Richmond Weather Bureau for the period 1905–1939 showed that rainfall during floodtime was no more, and no less, probable than at any other period. In 34 years the James River is normally above a stage of 8.0 ft for 4,080 hrs, 170 days, or 0.46 year. A stage of 8.0 ft was reported during each of 193 days in the period 1905–1939. Normally, rainfall of 0.01 in. or more is reported for 590 hr per yr, or 275 times in 170 days.

TABLE 3.—Normal Rainfall in Inches, Richmond, Va.

Maximum probable rainfall	Length of Period, in Days (Not Necessarily Consecutive)							
	1	2	5	10	20	50	100	200
Hourly...	0.03	0.07	0.14	0.23	0.35	0.56	0.76	0.98
Daily....	0	Trace	0.10	0.38	0.72	1.15	1.55	2.00

Table 3 shows the normal rainfall experience of Richmond. The quantities listed define the hourly and daily rainfall frequency curves from 1 to 200 days. (Note: The fact that hourly rainfall is greater than daily rainfall for short periods of time is correct, and merely means that, although rainfall of 0.01 in. or more occurs in only one of every 3 or 4 days, it usually rains for several hours when it does rain.)

The records of hourly and daily precipitation were compiled for all days, 1905–1939, inclusive, upon which a stage of 8.0 ft was reported, a total of 193 days. The precipitation for the entire day was included, regardless of whether the river remained above 8.0 ft. This inclusion is partly offset by the omission of records of days when the river has exceeded the stage for a few hours without having been reported.

Table 4(*a*) compares the normal experience of any 170 days at Richmond, selected at random, with the record of all days, 1905–1939, upon which the James River was reported above 8.0 ft. Table 4(*b*) shows a similar comparison for the experience above a stage of 20.0 ft, which is an average of 11 days in 34 years.

Tables 4(*a*) and 4(*b*) show essential agreement between the floodtime experience and that of a normal period of equal length. If the number of floodtime occurrences were 2.0 or 3.0 times the number of normal occurrences, the conclusion would be that floodtime rainfall was 2.0 or 3.0 times as probable as normal. In this case, there is no essential deviation, and the writer draws

the conclusion that the probability is the same, and that the estimate of rainfall at higher stages on a similar basis is warranted.

In this connection, it is well to note that it is not necessary to have a knowledge of the true duration of floods above given stages; but it is adequate

TABLE 4.—COMPARISON OF FLOODTIME AND NORMAL RAINFALL, RICHMOND, VA.; NO. OF OCCURRENCES

Excess (in.)	(a) RIVER ABOVE 8.0 FT				(b) RIVER ABOVE 20.0 FT			
	Hourly Rainfall		Daily Rainfall		Hourly Rainfall		Daily Rainfall	
	Floodtime[a] 1905 to 1939	Normal for 170 days	Floodtime[a] 1905 to 1939	Normal for 170 days	Floodtime 1905 to 1939	Normal for 11 days	Floodtime 1905 to 1939	Normal for 11 days
Trace	....	....	54	73	....	....	5	5
0.01	354	275	36	54	16	18	4	4
0.03	....	....	....	....	8	10	....	....
0.05	....	....	....	....	5	7	3	3
0.10	71	55	....	....	1	3	2	2
0.20	22	21	29	26	0	1	....	....
0.30	7	11	....	....	....	....	2	1
0.50	1	4	12	13	....	....	0	1
0.87	1	1	....	....	....	....	....	....
1.00	....	....	6	5	....	....	....	....
1.50	....	....	2	2	....	....	....	....
2.00	....	....	0	1	....	....	....	....

[a] During any part of a day on which a stage of 8.0 ft or higher was reported.

to use a flood-stage frequency curve with any assumed number of days' duration per rise. If the duration is assumed too long, the probability ratio will be correspondingly smaller. Likewise, a seasonal or monthly rainfall frequency curve could be used, instead of an annual curve as used by the writer, with a corresponding compensative change in the indicated relative probability of rainfall during floodtime and in the normal period for which a rainfall curve is available.

The writer believes that the theory of this method of analysis can be extended easily to runoff instead of rainfall. A pumping plant to be used above the 5-yr river stage, where average duration of floods is 2 days, is exposed to conditions under which it must operate for only 20 days in 50 years. One year of normal or near normal runoff would contain 18 such occurrences. Observations of runoff certainly should be made before any expensive final drainage plan is adopted.

W. W. HORNER,[15] M. AM. SOC. C. E.[15a]—Mr. Williams' paper deals with a class of engineering problems which appears to be occurring more and more frequently in engineering practice, and about which relatively little has been published. The title is somewhat too narrow, since the principles involved seem to apply clearly to the drainage of any relatively narrow levee-protected area where there is considerable lateral inflow from a tributary stream. For

[15] Cons. Engr. (Horner & Shifrin), St. Louis, Mo.

[15a] Received by the Secretary April 1, 1942.

example, the writer has encountered exactly the same conditions in connection with the "industrial property" area along the Trinity River at Dallas, Tex. At this point the lateral inflow was almost entirely from storm-sewered urban areas. He also has found such conditions in his work for the East Side Levee and Sanitary District at East St. Louis, Ill., where the lateral inflow is from small streams of 1 to 20 sq miles flowing out of the higher-level Illinois bluff drainage areas. Possibly by a little stretch of the imagination, these bluff lands might be considered as mountains. In the first instance, the only possible solution was through the use of pressure conduits; and in the second, cutoff channels, detention storage, and pumping are involved.

The procedure suggested by Mr. Williams is interesting, and in part new. It appears to justify some comments and a number of questions which the writer hopes that the author may be able to answer in his closing discussion. The two conditions referred to as (*a*) and (*b*) under "Local Drainage Requirements" are fundamental to all such problems. In the experience of the writer, item (*b*) is always the critical one and must be considered first, and item (*a*) should be taken up after the primary solution of item (*b*) has been determined.

Under the head of "Local Hydrology" Mr. Williams discusses the requirement for the consideration of complete hydrographs. This usually has not been undertaken in local problems of this kind, but it is fundamental to any reasonable solution. The one exception to this requirement is that of pressure conduits draining completely sewered urban areas where no storage is provided and the conduit capacity must be equivalent to the peak rate of flow from the storm sewers. For this situation, it is necessary to study the coincident probability of discharges of various frequencies with concordant stages in the main river.

In the case of the Dallas project (1931), a preliminary report had indicated pressure conduits of a capacity to carry the peak rate from the determined rainfall frequency of the urban storm system, in this instance a 5-yr frequency;

TABLE 5.—RESULTS OF LEVEES AND FLOODWAY CLEARANCE ON STAGE DURATIONS, TRINITY RIVER AT DALLAS, TEX.

Description	Flood Flow (Cu Ft per Sec):			
	75,000	63,000	49,000	23,000
Record Data (27 Years):				
Stage (ft)	42[a]	41	39	35
Days of above stage in 27 years	6	10	38	68
Rate of same excess with reservoirs	69,000	32,000	15,000	12,500
Stage, with cleared floodway between levees	39.5	35	33	30 ±

[a] Second high flood in 27 years; highest stage 52.6 ft.

but the preliminary design of the pressure conduits provided this capacity against the maximum recorded river stage in the Trinity River. This situation was quite critical to the economics of design, as the lower-lying portions of the

areas served by the storm sewers were but little above the maximum recorded flood stage.

This case was complicated by the then recent completion of reservoirs above Dallas, and by the changed régime involved in the construction of the levees and cleared floodway, and required a complex analysis of the effect of these structures on stage durations. The result of these studies is summarized in Table 5.

The required hydraulic grade to serve critical improvements in the vicinity of the head of the pressure conduit was at stage 54. The design stage at the outlet was fixed at 35, giving a drop of 19 ft in the 4,700 ft of pressure conduit. For this situation, an additional check also was made on the actual occurrence of rainfall equal to or exceeding the 1-yr frequency, with concurrent river stages. It was found that, although many of the precipitation periods were related to the same storm which produced the river rises, these precipitation periods never occurred at the passing of the peak of the flood, and were generally related to flood stages well down on the rising side of the hydrograph. It was found that, within the duration of stages in excess of 41 (or 35 with improvements), precipitation had occurred on only 1 day and in an amount of only 0.75 in.

Mr. Williams' experiences with basic data deficiencies closely parallel those of the writer, except that the writer generally has found more adequate recording rain-gage information than seemed to be available in this instance. The installation of stream gaging stations on small areas, such as that in Mr. Williams' project, seems to the writer to be a fundamental requirement for this class of work, the value of which has not been fully appreciated heretofore. Hydrologists are too prone to think that stream-flow records are of no value until a record of a number of years is available. Actually, a good record of two or three major rises of the hydrograph for a small watershed makes unnecessary reams of calculations in the field of synthetic hydrograph preparation, most of which do not lead to very convincing results.

The author's division of both rainfall and flood stages into two major seasons would seem to be adequate for the Susquehanna River. The writer used a similar division in the Dallas studies, determining rainfall depth-frequency relations for each of two 6-month periods; but because of climatic differences, the periods were not the same as those used by Mr. Williams. A study of rainfall frequency records for the winter seasons and the use of the results as indexes appear to be valid procedures for the project described. It is to be noted that the author does not propose a similar procedure for the summer season, and the writer agrees that the indexes for that season would not be sufficiently significant.

The use of point rainfall data for the complete drainage basin seems satisfactory for the winter season, but the writer would not utilize it for summer storms where the area is appreciably in excess of 1 sq mile.

Under the head of "Local Hydrology: Design Storm for Flood Season," the author struggles with a methodology that has been a cause of concern to all hydrologists. In the writer's experience, there is no fully satisfactory

answer to this requirement. The best and probably most valid answer involves not the choice of some severe storm, but the actual analysis of runoff for a large number of "severe" storms throughout the period of record, and the subjection of the results of a flood-frequency analysis. Something on this order was utilized in the writer's recent study of flood flow on the Trinity River for the U. S. Department of Agriculture. It involves a great amount of work, however—an amount that would be in excess of what normally would be undertaken for projects such as those described by the author, which involve a considerable number of small streams. The second method suggested, and the one which the author utilized in setting up a synthetic storm pattern from rainfalls of equal frequency, would seem to be definitely and probably considerably on the conservative side. The writer would hesitate to adopt it without further tests of the results. It unquestionably provides synthetic storm patterns of a much rarer and infrequent type than the chosen design frequency. A third method that sometimes has been utilized involves the setting up of a pattern for the design frequency precipitation in accordance with average occurrence. This involves a considerable analysis to determine average storm patterns, but extensive work in this field already has been done. The method was utilized by the Department of Agriculture in the Little Tallahatchie flood-control studies in Mississippi, and there are also available independent studies reported by D. I. Blumenstock.[16] The St. Louis regional rainfall records have been subjected to such studies under the supervision of the writer. Important conclusions which may be drawn from these studies are that:

(1) Such a design storm will never contain short-period intensities as great as the short-period intensities of equal frequency; and

(2) High rates will not occur at the midpoint in the storms, but more nearly at the first one-third point.

The effect of these differences is to give a somewhat flatter and somewhat more advanced histogram of rainfall than those shown by the author.

It is interesting that the author does not consider infiltration rates for the winter storms, possibly due to the fact that so little data have as yet been published for these conditions. The writer is inclined, however, to question the propriety of using the runoff as 100% of the precipitation, plus snow melt, in the design storm. It would seem that, except under very rare conditions, some infiltration and interception losses must be inevitable. The use of such a value in the maximum storm might be justified if the maximum storm were to occur in winter. The application of an infiltration "rate" of 0.3 in. per hr for summer storms, presumably of the short thunderstorm type, would seem to the writer to be overconservative. However, he is of the opinion that proper rate cannot be determined from stream flows on larger areas unless larger areas can be found that have approximately the same coverage of soil and vegetation. The only reasonable approach to the determination of a proper infiltration rate would be on the basis of studies of the identical soil

[16] *Technical Bulletin No. 698*, U. S. Dept. of Agriculture, Washington, D. C.

and cover involved in the particular drainage basin. Otherwise, considerable error in either direction may be involved.

The writer is interested in the approach to the development of a synthetic unit hydrograph. In the Trinity River investigation, the writer, in collaboration with Waldo E. Smith, M. Am. Soc. C. E., analyzed a large number of hydrographs at the Rockwall gaging station (821 sq miles). For this station relations were developed as follows:

$$\frac{Q_p}{Q} = 0.025 + 0.07\,(Q - 0.7) \qquad (5)$$

for values of Q, or mass surface runoff in excess of 0.7 in., and for Q_p, the peak-hour rate in inches per hour; also, from the recession curves,

$$q = K_s\,S \qquad (6)$$

in which q is in cubic feet per square mile and S is hour-second-feet per square mile. For Rockwall, the value of K_s was 0.046.

A study of smaller basins in the same physiographic province indicated that their relationship could be made a function of the basin area in square miles, as follows:

$$K_s = \frac{7.4}{A^{0.25}} \qquad (7)$$

for watersheds in excess of 20 sq miles. For 1 in. of runoff,

$$Q_p = \frac{1.11}{A^{0.6}} \qquad (8)$$

The base of the hydrograph in hours is

$$B = 2.5\,A^{0.6} \qquad (9)$$

and the lag time in hours from center of mass of excess rainfall to center of mass of the hydrograph is

$$L = 0.88\,A^{0.6} \qquad (10)$$

From these relationships it was possible to construct distribution graphs for basins of various areas. To these graphs excess rainfall was applied in time increments varying from 1 hr for basins of 23 sq miles or less to 6 hr for basins in excess of 200 sq miles.

These last four relationships (Eqs. 7 to 10) were most useful and were extremely helpful in working out the hydrograph form.

The use of the "time of concentration" as discussed by the author has not been found to be very satisfactory for natural drainage basins of this type, and the writer would prefer not to use a methodology that involved the so-called rational formula. He would like to call attention to the fact that the "*C*" in the rational formula involves both losses and delay factors. In the latter category it includes the effect of channel storage. If by some means a proper value of "*C*" has been determined, then it would seem that something like flood routing already has been done, and the hydrograph would include

channel-storage effects above whatever point it was applied. The writer is in agreement with the statement made immediately below Eq. 4 of the paper, but would go further and state that, in most instances, such an assumption is not only conservative but excessively conservative. In general, the writer is of the opinion that much better synthetic hydrographs can be developed from a study of even two or three actual hydrographs than through any approach involving time of concentration and equal time contours. In this connection he would call attention again to the importance of doing what was done under the described projects—namely, the installation of stream gaging stations at the point where the lateral streams enter the flood plain.

WALTER T. WILSON,[17] ASSOC. M. AM. SOC. C. E.[17a]—The writer likes Mr. Williams' analytical approach, that is, studying floods by examining their causes. The logic and merit of studying floods by relating stream flow to the antecedent and causative rainfall cannot be denied. Without wishing to detract from Mr. Williams' excellent paper, the writer, feels that the study of successive causes and effects leading to floods should be carried at least one step nearer the first cause. The Hydrometeorological Section of the Weather Bureau has been studying the causes of major storms.

The expression "maximum probable" flood seems unsatisfactory because the term "maximum" implies an upper limit, whereas the term "probable" involves a wide range of meaning and begs a specific definition. A flood or a storm must have a finite physical upper limit. The writer, with an increasing number of hydrologists and meteorologists, is perfectly willing to "stick his neck out," and study this thing of which all hydrologists are thinking—the "maximum possible" storm or flood.

Storm magnitude is determined and limited by the nature and magnitude of its causes. These include the available quantity of moisture in the air, the rate at which moisture laden air flows into the flood area, and the rate at which the moisture can be condensed and precipitated. These factors have finite limits, which involve an admittedly difficult but not hopelessly impossible study.

By way of adverse criticism, the writer feels that the subject of snow is passed by with an inadequacy both in its treatment and melting rate. Mr. Williams' estimated design rate of an inch of snow melt per day has an obscure basis and seems too low. This is in contrast not only to reasonable degree-day factors but also in contrast to actual records. Reference is made to the 1936 flood in New England. The total runoff for the second storm period (March 16 to 19) from the basin of the East Branch of the Pemigewasset River, above Lincoln, N. H. (104 sq miles), has been estimated at 12 in. to 15 in. The nature of the topography and lack of good precipitation data require estimation of the precipitation during this period, which by preparation of isohyetal maps and careful consideration of meteorological factors indicates a value of 4-in. to 5-in. average depth. The difference between the 12 in. to 15 in. of runoff and the 4 in. to 5 in. of rain probably includes some ground-water con-

[17] Asst. Hydrologic Engr., Weather Bureau, Washington, D. C.

[17a] Received by the Secretary April 16, 1942.

tribution, but leaves about 8 in. of snow. The writer believes that at least 4 in. of snow melted and ran off in less than one day. By assuming a reasonable velocity of concentration and a reasonable ratio of stream length to basin area, Eq. 4 "boils down" to a Myer's rating of about 30% to 50%.

Under the heading "Local Drainage Requirements," the author states that flood protection from small tributaries is rarely justified. The writer realizes the danger of separating one sentence from the remainder of the text, and may not understand the meaning in this case. However, Mr. Williams has exposed himself to criticism by proponents of upstream engineering and flood control in source regions.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

ALLOCATION OF THE TENNESSEE VALLEY AUTHORITY PROJECTS

Discussion

BY EUGENE L. GRANT, M. AM. SOC. C. E.

EUGENE L. GRANT,[5] M. AM. SOC. C. E.[5a]—Once a multiple-purpose project is completed, it may be operated in various ways. The demands of flood control for empty reservoirs and of power for full reservoirs are particularly in conflict. If a project really is operated with flood control as a primary objective and power a secondary one, the firm power generated may be only a small fraction of that obtainable with operation for power as a primary or coordinate objective. This may be true even though the method of operation may not have much effect on the average annual kilowatt-hours of energy generated.

The alternative-justifiable-expenditure method of cost allocation must necessarily be based on some assumed scheme of reservoir operation. This, in turn, determines the alternative plant and the alternative cost. Thus different prospective plans of operation will involve some differences in the costs allocated among the various purposes. However, it seems probable that—in some multiple-purpose projects, at least—the differences in benefits to the several purposes of the project with various plans of operation will be much greater than the differences in costs allocated to these purposes. This may be particularly evident where the value of firm power from one plan of operation is compared with the value of surplus power from some other plan. It would be interesting if, in his closure, Mr. Parker would comment on this point in relation to the TVA cost allocation study. Were cost allocations made for more than one plan of operation? If so, were allocated costs compared with benefits, and particularly power costs with power values, under the various plans?

Whatever the prospective plan of operation adopted in the design of a multiple-purpose project, it seems likely that circumstances will operate to

NOTE.—This paper by Theodore B. Parker, M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Malcolm Elliott, M. Am. Soc. C. E.; and March, 1942, by E. L. Chandler, M. Am. Soc. C. E.

[5] Prof. of Economics of Eng., Stanford Univ., Dept. of Civ. Eng., Stanford University, Calif.

[5a] Received by the Secretary April 14, 1942.

cause departures from this plan. For instance, under war-time conditions of power shortage in the Tennessee Valley, it would seem reasonable to give major consideration to power in operating TVA reservoirs. In more normal times, the pressures on the operators of government multiple-purpose projects are likely to be strongest from those beneficiaries of the projects who make no special contribution to their costs—that is, from flood control and navigation interests. In this connection the propensity of congress to designate multiple-purpose projects as primarily for flood control and navigation regardless of their real justification may prove to be unfortunate. The Central Valley project in California, conceived as an irrigation project to be financed in large part from sale of power, but designated by congress as primarily a navigation and flood control project, is a case in point.

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DISCUSSIONS

DESIGN OF SIGN LETTER SIZES

Discussion

By Guy Kelcey, M. Am. Soc. C. E.

Guy Kelcey,[17] M. Am. Soc. C. E.[17a]—In their presentation of this too little explored subject, Messrs. Mitchell and Forbes have made a definite contribution in the direction of more efficient and, therefore, of safer roadway usage. Although the subject presented may be of quite limited interest to many engineers, it is of great importance to those who are responsible for best use of roadways for purposes of transportation.

A traffic sign has two values. A driver's attention is attracted to a sign, before he is within its legibility range, by its target value. At closer approach he is able to read its message or to perceive its symbol. If sign copy is too small to be useful to high-speed traffic, except at very close range, special attention must be given to the target value of such signs by the use of conspicuous background colors and shapes, such as octagon, round, square, or diamond, so that the target characteristic of the sign will tell drivers from a distance what general type of road condition is ahead. When, as Messrs. Mitchell and Forbes suggest, sign copy is enlarged so that its legibility has adequate distance value, in terms of the present-day speed characteristics of the location, target value, beyond that provided by the copy or symbol, becomes of lesser or of no importance.

This leads to an interesting conclusion. In the past, traffic sign design has tended to ignore legibility value and copy was condensed into standard sized target space. Pursuit of the principles presented by Messrs. Mitchell and Forbes would reverse this procedure. It would result in designing to legibility value and then adapting the size and shape of the sign background to the resulting copy dimensions. Since the copy or symbol would be large enough to provide any target value needed, no great attention need be paid to providing target value in the sign background. If so, the most conspicuous colors could be used in the copy or symbol and even greater legibility values achieved.

Note.—This paper by Adolphus Mitchell, Assoc. M. Am. Soc. C. E., and T. W. Forbes, Esq., was published in January, 1942, *Proceedings*.

[17] Cons. Engr., Motor Transport and Traffic; Pres., Vehicular Parking, Ltd., Newark, N. J.

[17a] Received by the Secretary April 10, 1942.

The practice of placing traffic signs in advance of the point to which their message applies is regarded by some as a needless carry-over from conditions of many years ago. Painted signs were originally placed in advance of hazards largely because they could not be read at night until a driver was close; he then needed some distance—generally 300 ft—in which to act. This practice of advance location of traffic signs lost its major purpose with the introduction of reflectors and other means to illuminate copy for legibility from a distance after dark.

Just as a lantern is best placed directly on a pile of brick in a street and not 300 ft in advance, there is much to be said in favor of placing a luminous or illuminated warning or information sign at, or close to, the point to which its message applies. Thus, the message is not only projected to drivers at a distance, but the drivers are brought all the way up to the point to which the message applies.

It is generally overlooked, too, that the location of signs well in advance of a hazard often introduces a danger. Like a structural worker walking high steel, a good driver looks up and not down. His gaze is normally focused several hundred feet or more ahead. When a traffic sign, with copy legibility of from 200 to 300 ft, comes into view, a driver reads its message from or near that distance and the focus of his attention sweeps on. If this distance is added to the 300 ft that the sign is placed in advance of the danger, it will be noted that it is possible for a driver, whose attention is focused on the road in front, to travel the last 500 ft or more with no further reminders of the condition ahead unless the danger itself is conspicuous. At night a driver can and often does forget the message on signs placed in advance of danger points, and being unable to see the hazard ahead, is taken by surprise.

Messrs. Mitchell and Forbes present a method by which sign copy size requirements may be determined accurately for each sign location, thus to provide adequate legibility values to satisfy determined conditions. In practice it should be possible to establish a series of copy sizes in 3-in. or in 6-in. steps and to use the nearest suitable size. Tailor-made traffic signs, with copy exactly proportioned to variable legibility distance needs, could result in a multiplicity of copy dimensions that would increase the cost of manufacture and of inventory and complicate installation and maintenance.

Free space available adjacent to the traveled part of roadways and streets is often so limited that signs, larger than those now used, could not always be accommodated if installed in accordance with present standards. If legibility values greater than those provided by currently standard sign copy are to be employed, it will be recognized that much larger traffic signs would be used. If so, in many cases, they must either be suspended over the roadway or bracketed over the shoulder or the sidewalk. Indeed this has already been done at scattered locations in some jurisdictions and results are reported to be excellent.

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DISCUSSIONS

REDUCTION OF MINERAL CONTENT IN WATER WITH ORGANIC ZEOLITES

Discussion

BY HARVEY F. LUDWIG AND RUSSELL G. LUDWIG, JUNIORS, AM. SOC. C. E.

HARVEY F. LUDWIG[9] AND RUSSELL G. LUDWIG,[10] JUNIORS, AM. SOC. C. E.[10a]—An interesting review of what may be accomplished in the water conditioning field, through the use of various modern zeolites, is contained in this paper. Mr. Goudey does not elaborate upon the basic mechanism by which these zeolites perform their work—namely, that of "ion or adsorption exchange"—and the writers believe that a brief but critical analysis of this exchange phenomenon would be of considerable interest. This basic mechanism has wide and important, but generally unrecognized, application in many engineering fields, particularly in sanitary engineering.

Colloid chemists have determined that the ion exchange properties of zeolitic material in aqueous suspension result from the existence of an electrical double layer surrounding the nucleus or core of each particle. At the surface of the core reside relatively fixed charges or ions ("stabilizing ions") of either positive or negative charge; these make up the inner part of the double layer. The outer section comprises an envelope of oscillating ions ("adsorbed ions" or "exchangeable ions" or "replaceable ions") of opposite charge which diffuses into the surrounding dispersion medium (water). In the ordinary hardness-removing zeolite, for example, each particle (nucleus) is negatively charged, and this is balanced by an outer layer consisting largely, in freshly regenerated zeolites, of sodium ions (Na^+). These adsorbed Na^+ ions are highly dissociated—that is, they oscillate relatively far into the dispersion medium because they are weakly adsorbed.

The degree to which a particular ion will be held or adsorbed by the inner layer may be termed its "exchange adsorbability." The magnitude of this

NOTE.—This paper by R. F. Goudey, M. Am. Soc. C. E., was published in February, 1942, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by Messrs. James M. Montgomery, and George L. Davenport, Jr.; and April, 1942, by Robert Spurr Weston, M. Am. Soc. C. E.

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[10a] Received by the Secretary March 30, 1942.

varies inversely with the hydrodynamic size of the ion, and directly with its valence. Ions in aqueous solution are thought to be hydrated or surrounded by a water shell of diameter inversely proportional to the actual size of the ion as it occurs in the crystal lattice. The combined effect of valence and of hydrodynamic size has been expressed[11] by the term "oscillation volume," representing the effective volume occupied by the oscillating ion, or, in other words, the width of the double layer. The smaller the oscillation volume, the more strongly an adsorbed ion is held and the greater its exchange adsorbability. Thus, the cations commonly found in natural waters may be classified in increasing order of their exchange adsorbability as follows: Na^+, K^+, Mg^{++}, Ca^{++}, H^+, Fe^{+++} and Al^{+++}. The trivalent Al^{+++} and Fe^{+++}, when they exist as such, are strongly adsorbed. The action of the H^+ ion is anomalous, but usually it behaves like a weak trivalent cation. The anions may be similarly classified, but in general much less is known of their behavior than of the cations.

If in the dispersion medium there are ions other than those of the diffused outer layer, then the phenomenon of "adsorption exchange" will occur, in which ions initially present in the outer layer are displaced by those from the dispersion medium. The degree to which one ion will displace another will depend upon the relative exchange adsorbabilities and upon the relative concentrations, in accordance with the general mass law. Thus, in the softening of water with ordinary zeolites, the hardness cations, Ca^{++} and Mg^{++}, displace the less strongly adsorbed Na^+ that is initially present in the zeolite and are thereby removed from solution. In the process of regeneration, the weakly adsorbed Na^+ is made to displace Ca^{++} and Mg^{++} by leaching the zeolite with a great preponderance of Na^+. Occasionally some Al^{+++} or Fe^{+++} or H^+ are also adsorbed in the softening operations. These are so strongly held that they may not be appreciably displaced by regeneration, and in time the zeolite "wears out." As noted by Mr. Goudey, wearing out may also occur through disintegration of the zeolite structure in the presence of excessive H^+ or OH^-.

Flocculation of Turbid Waters.—The mechanism of adsorption exchange has a multitude of other applications of interest to the civil engineer. In the flocculation of turbid waters, for example, the phenomenon is particularly important, in that the stability of suspended turbidity (clay) particles results largely from their zeolitic properties. Each particle is surrounded by an electric double layer, the outer section of which consists of various cations (combinations of Na^+, K^+, Ca^{++}, Mg^{++}, and H^+) of positive charge. The particles therefore repel each other, resisting their natural tendency to aggregate; and the forces of repulsion are proportional to the effective width of the double layer. When alum (or a similar coagulant) is added to the system, a portion of this dissociates to form Al^{+++} ions. These replace the cations initially adsorbed and thus reduce the effective width of the double layer to a critical value at which the particles no longer repel each other but instead tend to agglomerate. Another portion of the added alum is converted by

[11] "Properties of Colloids," by Hans Jenny, Stanford Univ. Press, Stanford University, Calif., 1938.

hydrolysis (chemical combination with water) to particles or micelles of hydrous aluminum oxide. These are excellent binder material, and when the system is properly mixed or agitated they serve to entrap or "wrap up" the suspended turbidity particles and thereby form large-sized, dense flocs, which will settle out quickly on subsequent standing. The ion exchange reactions are important in this process because by this mechanism the initially stable turbidity particles are rendered unstable, so that they may be entrapped easily by the agglomerating hydrous oxide floc. The "coagulant demand" or dosage of coagulant necessary to flocculate a given water therefore may be considered as comprising two parts—a "perikinetic demand" or that part of the alum which dissociates to form Al^{+++}, and a "binder demand" or that alum which hydrolyzes to give hydrous oxide. (The term "perikinetic" refers to the forces of attraction and repulsion associated with the suspended turbidity particles. The action of Al^{+++} on the system reduces the forces of repulsion such that the particles will tend to aggregate. The system is now "perikinetically flocculated.") The "optimum pH" for any turbid water will be that at which the added alum will dissociate and hydrolyze just as required by the relative magnitudes of the perikinetic and binder demands. These concepts of the mechanics of flocculation were originally presented by the writers in a previous publication.[12] Their studies are being continued in the Sanitary Engineering Laboratories of the University of California, at Berkeley, under the direction of Prof. W. F. Langelier.

Soil Corrosion.—Another important application of the ion exchange principle is found in the study of soil corrosion of pipes. The basic mechanism involved has been termed "contact exchange," and may be demonstrated by hanging an iron nail in a suspension of clay particles containing replaceable H^+ ions. In time it is found that the nail may disappear entirely, and that the clay colloids have acquired a brownish coloration. Evidently the H^+ ions of the clay have been displaced by iron from the nail. This phenomenon is undoubtedly important in the corrosion of metallic pipes laid in soils containing replaceable hydrogen. Similarly, the deterioration of concrete pipe laid in soils containing adsorbed Mg^{++} properly may be attributed to replacement of Ca^{++} by Mg^{++}.

The question arises here as to just what adsorbed cations might be expected to occur in the soils of a given geographical region. Both the total concentration of adsorbed cations (called the "saturation capacity" or "base exchange") and the specific types present will depend largely upon climate, particularly upon annual rainfall.[13] The adsorbed cations in soils are associated with those soil particles of colloidal size (about 1 μ and less), termed the soil colloids; these include both the inorganic colloids (clay) and the organic humus colloids. The adsorbed cations generally include Na^+, K^+, Ca^{++}, Mg^{++}, and H^+; and the saturation capacity is usually expressed as the sum of the exchangeable bases (Na^+, K^+, Ca^{++}, Mg^{++}) plus the exchangeable hydrogen (H^+), all

[12] "Laboratory Flocculation of East Bay Sewage, and the Mechanism of Flocculation in Water Purification and Sewage Treatment Practice," by H. F. Ludwig and R. G. Ludwig, *California Sewage Works Journal*, Vol. 13 (1941), pp. 54–80.

[13] "Factors of Soil Formation," by Hans Jenny, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

measured in milligram equivalents per 100 grams of soil. In arid regions the soils (alkali soils) contain relatively few colloids and therefore have low saturation capacities; the replaceable cations are all exchangeable bases (about 50% Na^+ and K^+, and about 50% Ca^{++} and Mg^{++}), there being no exchangeable hydrogen. When the annual rainfall becomes appreciable (in the magnitude of 25 in.), the leaching action of the percolating water results in replacement of some of the exchangeable bases by H^+ ions derived from the free carbonic acid in the water that percolates through the soil. Simultaneously, there is an increase in the saturation capacity, because the increased percolation hastens the weathering processes by which the noncolloidal primary soil minerals are converted into colloidal particles (clay), and because the increase in vegetation associated with increased rainfall increases the concentration of the humus colloids. In such soils (chernozems) the adsorbed cations comprise from 1% to 10% each of Na^+, K^+, Mg^{++}, and H^+, with about 75% Ca^{++}. Finally, in the very humid regions, the soils (podsols) are of very high saturation capacity and comprise about 70% H^+ with the remainder mostly Ca^{++} and Mg^{++}, there being little or no Na^+ or K^+.

It is interesting to note that the physical and chemical properties of soils are generally controlled by the magnitude and nature of the soil colloids, although these usually are only a fraction of the total soil mass (about 10% in ordinary loams, for example). Further, the properties of these colloids are largely determined by the concentration and nature of their exchangeable cations, and these represent only about 5% of the soil colloid mass. Thus, the properties of a soil are determined essentially by its exchangeable cation content, which generally comprises less than 1% of the entire soil mass.

These two examples have served to illustrate how prevalent and important ion exchange reactions may be. There are many other instances, as, for example, in the fields of sewage treatment, highway construction, irrigation, etc.

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DISCUSSIONS

DESIGN OF ST. GEORGES TIED ARCH SPAN

Discussion

BY ALEXANDER DODGE, ESQ.

ALEXANDER DODGE,[6] ESQ.[6a]—Professor Garrelts undertakes to design a multiply indeterminate structure by a simplified method. A design of this nature is usually very cumbersome, and in practice various short cuts frequently are permitted. However, the introduction of various assumptions by the author, without proof of their correctness, has made his paper inconclusive. The disputable parts of the analysis are as follows:

1. In the preliminary analysis, by assuming that the arch rib has resisted no bending, as well as by omitting the term $\sum \frac{M''_R \, m \, \Delta l}{E \, I_R}$ from Eq. 4, the author assumes that the rib is composed of a number of members pin-connected at the springing and at each hanger point. How the condition of continuity in the structure was finally restored has not been shown.

2. To find the distribution of bending moment between arch rib and girder, their relative deflections were supposed to be used, but the differential equation (Eq. 10*a*) holds true only for a thin curved bar of a constant cross section with circular axis and when subjected to transverse loading only.[7] A small initial curvature makes a great change in the effect of longitudinal forces on the deflection of the arch rib. The correct differential equation for deflection of an arch rib has been published by S. Timoshenko.[8]

The paper does not give full information regarding properties of the structure such as cross section and moment of inertia of the arch rib and the tie girder. Nevertheless, Figs. 6 and 7 give some indications on which the writer bases the following deductions:

The distribution of moment of inertia in an arch rib is probably not very different from that commonly used in practice, variation being proportional to

NOTE.—This paper by J. M. Garrelts, Assoc. M. Am. Soc. C. E., was published in December, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1942, by R. W. Abbett, M. Am. Soc. C. E.; and April, 1942, by Jacob Karol, Esq.

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[6a] Received by the Secretary April 13, 1942.

[7] "Strength of Materials," by S. Timoshenko, D. Van Nostrand, 1930, Pt. II, p. 457.

[8] *Ibid.*, pp. 460–467; see also, "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, pp. 230–236.

sec α. The average area of tie used in computing deformations is probably about 265 sq in., and the moment of inertia at the crown, I_C, about 48,000 in.[4] These values will be used in the discussion that follows, as well as the given ratio of tie girder to arch rib (13 : 1). The table of ordinates in Fig. 6 shows that the arch rib is parabolic in shape with a ratio of $\frac{H}{L} = 0.185$.

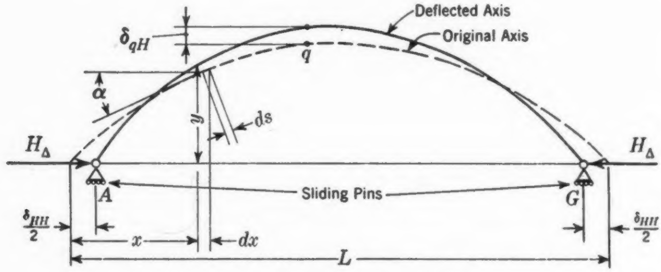


Fig. 9.—Isolated Arch Rib

Assuming a pin condition at the springing and neglecting rib shortening, the deflection of an arch rib (Fig. 9), at a point q due to H_Δ, is

$$\delta_{qH} = \frac{H_\Delta}{E} \int_A^G y \frac{ds}{I} m' = \frac{H_\Delta}{E\, I_C} \int_A^G y\, m'\, dx = \frac{H_\Delta\, h\, L^2}{E\, I_C}\, \epsilon \ldots\ldots (28)$$

In Eq. 28, $y = \frac{4\,h}{L}\left(x - \frac{x^2}{L}\right)$; $I = I_C \sec \alpha$; I_C = minimum moment of

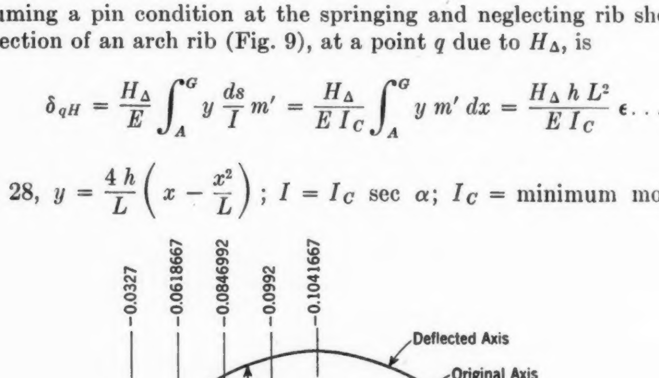


Fig. 10.—Coefficients of Vertical Deflections, ϵ, Due to $H = 1$

inertia (at the crown); m' = the simple beam bending moment in the arch rib due to a unit vertical load at point q, when $H = 0$; and ϵ = the numerical coefficient of the vertical deflection.

By Maxwell's Law of Reciprocal Deflections δ_{Hq} = the horizontal deflection at A due to a vertical load P at $q = \delta_{qH}$, or $\delta_{Hq} = \delta_{qH} = \frac{H_\Delta\, h\, L^2}{E\, I_C}\,\epsilon = \frac{P\, h\, L^2}{E\, I_C}\,\epsilon$ provided the horizontal load H_Δ is equal to P.

The horizontal deflection δ_{HH} of the point A due to H_Δ is

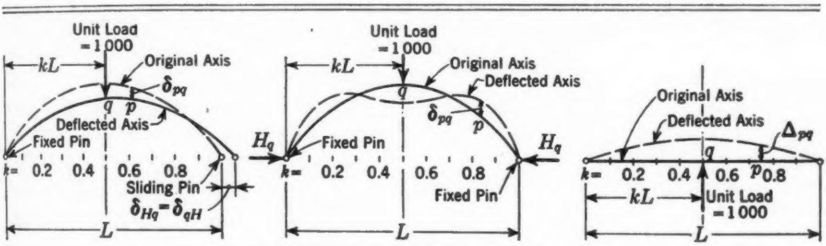
$$\delta_{HH} = \frac{H_\Delta}{E}\int_A^G \frac{m_H\, m_H\, ds}{I} = 0.53333\,\frac{h^2\, L\, H_\Delta}{E\, I_C} \qquad \text{(29)}$$

in which m_H = the simple beam bending moment in the arch rib due to a unit horizontal load at A.

The deflection of the arch rib at p due to a unit vertical load at q is

$$\delta_{pq} = \int_A^G \frac{M_S\, m'\, ds}{E\, I} - H_q \int_A^G \frac{m_H\, m'\, ds}{E\, I} = \frac{L^3}{E\, I_C}(\lambda - \phi) = \frac{L^3}{E\, I_C}\,\eta \qquad \text{(30)}$$

TABLE 2.—Coefficients of Vertical and Horizontal Deflections Due to a Unit Vertical Load of 1,000



(Note: Downward deflections positive)

Position of load (tenth point)	Coefficients of Vertical Deflections, for k Equal to: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Coefficients of horizontal deflection
	(*a*) Values of λ									Values of 1,000 ϵ
0.5	6.1667	11.8333	16.5000	19.6667	20.8333	19.6667	16.5000	11.8333	6.1667	104.1667
0.6	5.5333	10.6667	15.0000	18.1333	19.6667	19.2000	16.5000	12.0000	6.3000	99.2000
0.7	4.5000	8.7000	12.3000	15.0000	16.5000	16.5000	14.7000	10.9667	5.8333	84.6992
0.8	3.1667	6.1333	8.7000	10.6667	11.8333	12.0000	10.9667	8.5333	4.6667	61.8667
0.9	1.6333	3.1667	4.5000	5.5333	6.1667	6.3000	5.8333	4.6667	2.7000	32.7000
	(*b*) Values of η									
0.5	−0.2200	−0.2500	−0.0430	+0.2917	+0.4882	+0.2917	−0.0430	−0.2500	−0.2200	....
0.6	−0.5489	−0.8405	−0.7542	−0.3179	+0.2917	+0.7488	+0.7458	+0.4928	+0.2178	....
0.7	−0.6932	−1.1252	−1.1514	−0.7542	−0.0430	+0.7458	+1.2486	+1.1415	+0.6401	....
0.8	−0.6265	−1.0432	−1.1252	−0.8405	−0.2500	+0.4928	+1.1415	+1.3568	+0.8735	....
0.9	−0.3716	−0.6265	−0.6932	−0.5489	−0.2200	+0.2178	+0.6401	+0.8735	+0.6951	....
	(*c*) Values of C									
0.5	−0.474	−0.910	−1.269	−1.513	−1.603	−1.513	−1.269	−0.910	−0.474	....
0.6	−0.426	−0.821	−1.154	−1.395	−1.513	−1.477	−1.269	−0.923	−0.485	....
0.7	−0.346	−0.669	−0.946	−1.154	−1.269	−1.269	−1.131	−0.844	−0.449	....
0.8	−0.244	−0.472	−0.669	−0.821	−0.910	−0.923	−0.844	−0.656	−0.359	....
0.9	−0.126	−0.244	−0.346	−0.426	−0.474	−0.485	−0.449	−0.359	−0.208	....

in which: M_S = the simple beam moment in the arch rib due to a unit vertical load at q; H_q = the horizontal reaction at the springing due to a unit vertical

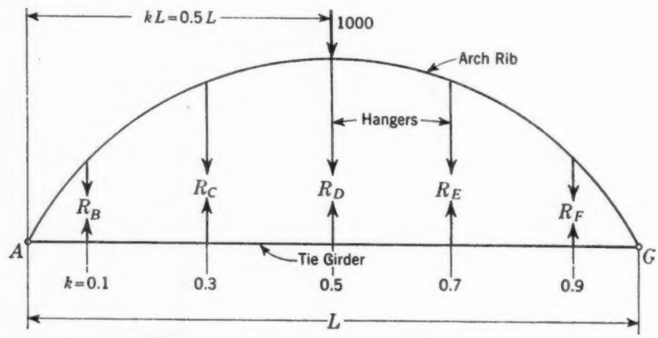


Fig. 11.—Tied Arch

load at q, for an arch, with fixed pins to prevent horizontal movements; λ = the numerical coefficient of the vertical deflections for an arch acting as a simple beam; ϕ = the numerical coefficient of the vertical deflection due to H_q alone = $H_q\,\epsilon$; and η = the numerical coefficient of the final vertical deflection. Coefficients ϵ, λ, and η are obtained by performing the necessary integrations. The coefficients of deflections were computed by the writer and are recorded in Fig. 10 and Table 2. The inspection of these coefficients will indicate immediately the importance of the longitudinal force (horizontal thrust) on the deflection of the arch rib, as has been stated previously in this discussion.

TABLE 3.—Summary of Deflections Due to Vertical Loads

Unit external load at k	(a) Arch Rib Deflections				
0.5	δ_B	δ_C	δ_D	δ_E	δ_F
Unit interaction load at k	(b) Arch Rib Deflections				
0.1	δ_{BB}	δ_{CB}	δ_{DB}	δ_{EB}	δ_{FB}
0.3	δ_{BC}	δ_{CC}	δ_{DC}	δ_{EC}	δ_{FC}
0.5	δ_{BD}	δ_{CD}	δ_{DD}	δ_{ED}	δ_{FD}
0.7	δ_{BE}	δ_{CE}	δ_{DE}	δ_{EE}	δ_{FE}
0.9	δ_{BF}	δ_{CF}	δ_{DF}	δ_{EF}	δ_{FF}
Unit interaction load at k	(c) Tie Girder Deflections				
0.1	Δ_{BB}	Δ_{CB}	Δ_{DB}	Δ_{EB}	Δ_{FB}
0.3	Δ_{BC}	Δ_{CC}	Δ_{DC}	Δ_{EC}	Δ_{FC}
0.5	Δ_{BD}	Δ_{CD}	Δ_{DD}	Δ_{ED}	Δ_{FD}
0.7	Δ_{BE}	Δ_{CE}	Δ_{DE}	Δ_{EE}	Δ_{FE}
0.9	Δ_{BF}	Δ_{CF}	Δ_{DF}	Δ_{EF}	Δ_{FF}

Deflection of the tie girder, acting as a simple beam, due to a unit vertical load is

$$\Delta = \frac{C'\,L^3}{E\,I_G} = \frac{C\,L^3}{E\,I_C} \quad \ldots (31)$$

in which C = the coefficient of the vertical deflection $= \frac{C'}{13}$ (see Table 2(c)). The ratio $\frac{I_G}{I_C} = 13$ is assumed to hold true at the crown.

The behavior of the St. Georges Tied Arch has been studied by means of coefficients of deflections on the assumption that five hangers only were acting, each hanger having a gross area of 35 sq in. The unknowns are a horizontal thrust, H, and five hanger interactions, assumed to be all tensions, as shown in Fig. 11. The summary of deflections for these conditions is given in Table 3. The subscript notations in this table refer—first letter, to the point on the structure, and second letter, if shown, to the point at which a force is acting. For example, δ_{BD} is a deflection of the arch rib at point B due to a load acting at point D.

For a unit load of 1,000 lb at $k = 0.5$, the five deflection equations in terms of deflection coefficients (the term $\dfrac{L^3}{E\,I_C}$ is dropped from all following computations) for points, B, C, and D, respectively, will be as follows:

$$- 0.220 + 0.6951\,R_B + 0.6401\,R_C - 0.220\,R_D - 0.6932\,R_E - 0.3716\,R_F + 0.208\,R_B + 0.449\,R_C + 0.474\,R_D + 0.346\,R_E + 0.126\,R_F = 0 \quad (32a)$$

$$- 0.043 + 0.6401\,R_B + 1.2486\,R_C - 0.043\,R_D - 1.1514\,R_E - 0.6932\,R_F + 0.449\,R_B + 1.131\,R_C + 1.269\,R_D + 0.946\,R_E + 0.346\,R_F = 0 \quad (32b)$$

and

$$+ 0.4882 - 0.220\,R_B - 0.043\,R_C + 0.4882\,R_D - 0.043\,R_E - 0.220\,R_F + 0.474\,R_B + 1.269\,R_C + 1.603\,R_D + 1.269\,R_E + 0.474\,R_F = 0 \quad (32c)$$

By the law of symmetry $R_B = R_F$ and $R_C = R_E$. Solution of these equations yields (in pounds):

$$R_B = (0.2467)\,(1{,}000) = 247$$
$$R_C = (0.2983)\,(1{,}000) = 298$$
$$R_D = (-\,0.6432)\,(1{,}000) = -\,643$$

Taking these interactions into consideration, the horizontal thrust $H = 1{,}052$ lb. The coefficients of true deflections are $-\,0.015$, 0.000, and $+\,0.040$ for points B, C, and D, respectively.

Due to $H = 1$, the elongation of the tie girder is $\tau_{HH} = \dfrac{L}{A\,E}$. The horizontal movement of the arch rib at the springing is $\delta_{HH} = \dfrac{0.53333\,h^2\,L}{E\,I_C}$, and $\dfrac{\tau_{HH}}{\delta_{HH}} = \dfrac{I_C}{0.53333\,A\,h^2} = \dfrac{48{,}000}{(0.53333)\,(265)\,(1{,}200)^2} = 0.000236$. Therefore, the horizontal thrust that would exist, were the arch rib fixed against the horizontal movement, will be reduced, due to stretch in the girder, only 0.0236 of 1%.

For various positions of the 1,000-lb load, the reduction in horizontal thrust, H_τ, due to the elongation of the tie girder, is as follows:

k	H_τ (lb)
0.5	$1{,}055 \times 0.000236 = 0.249$
0.3	$857 \times 0.000236 = 0.202$
0.1	$331 \times 0.000236 = 0.078$

Taking the effect of this reduction into account, and with the aid of Fig. 10, Eqs. 32 will be changed to:

$$0.6627\ R_B + 0.7551\ R_C + 0.2621\ R_D - 0.220 = 0 \qquad (33a)$$

$$0.7551\ R_B + 2.2084\ R_C + 1.2471\ R_D - 0.043 = 0 \qquad (33b)$$

and

$$0.5242\ R_B + 2.4942\ R_C + 2.1169\ R_D + 0.4882 = 0 \qquad (33c)$$

The solution of these equations yields practically the same values of interactions as have been found previously in this discussion. Hanger elongations similarly had been incorporated in deflection equations, and the effect of these deformations has been found to be even smaller than the effect of the tie girder stretch.

So far, the girder has been assumed to be perfectly straight, but actually it follows the vertical curve of the roadway. The deformations due to this initial parabolic curvature may be taken into account as follows: Due to $H = 1$, the moment in the tie girder is equal to

$$y' - \Delta = \frac{4\ r}{L}\left(x - \frac{x^2}{L}\right) - \Delta \qquad (34)$$

and the differential equation of deflection is

$$\frac{d^2\Delta}{dx^2} = \frac{4\ r}{E\ I_G\ L}\left(x - \frac{x^2}{L}\right) \qquad (35)$$

Therefore,

$$\Delta = \frac{r\ L^2}{3\ E\ I_G}(-\ k + 2\ k^2 - k^4) = \frac{L^3}{E\ I_C}\ C'' \qquad (36)$$

in which r = maximum vertical ordinate of the tie girder = 6.8 ft; Δ = final deflection of the girder, which, being obviously insignificant, has been omitted from the right side of Eq. 35; and C'' = the numerical coefficient of deflection, expressed by

$$C'' = \frac{r\ (-\ k + 2\ k^3 - k^4)}{(3\ L)\ (13)} \qquad (37)$$

These equations are written with the origin at the springing. Coefficients C'' are recorded in Fig. 12. By means of these coefficients and Fig. 10, the effect

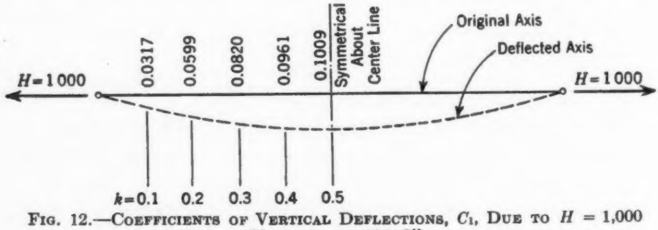


FIG. 12.—COEFFICIENTS OF VERTICAL DEFLECTIONS, C_1, DUE TO H = 1,000 (Note: C_1 = 1,000 C'')

of the tie girder deflections due to a horizontal thrust is easily incorporated into the analysis, and Eqs. 33 are changed to:

$$0.6417\ R_B + 0.7007\ R_C + 0.2287\ R_D - 0.220 = 0 \qquad (38a)$$

$$0.7007\ R_B + 2.0678\ R_C + 1.1606\ R_D - 0.043 = 0 \qquad (38b)$$

and

$$0.4574\ R_B + 2.3212\ R_C + 2.0105\ R_D + 0.4882 = 0 \ldots\ldots (38c)$$

The solution of Eqs. 38 indicates no material changes in the magnitude of interactions that have been determined previously.

Two other factors—interaction moment at the springing and rib shortening—similarly could be taken into consideration in the analysis. (The writer leaves it to students of arch design to prove that the interaction moment at the springing has an insignificant influence on design of this tied arch.) As it stands, the arch rib is considered as being pin-connected to the tie girder. The rib shortening, if taken into consideration, will reduce the value of the horizontal thrust. This in turn would increase the discrepancy between the writer's and the author's H-values.

Similarly, a new set of deflection equations has been made for interactions at every tenth point of the span. The summary of results from this study, for a unit vertical load of 1,000 applied to the girder at $k = 0.5$, is recorded in Table 4.

TABLE 4.—SUMMARY OF INTERACTION FORCES, MOMENT RATIOS, AND COEFFICIENTS OF DEFLECTION

(Horizontal Thrust = 1,054 Lb)

Description	$k = 0.1$	$k = 0.2$	$k = 0.3$	$k = 0.4$	$k = 0.5$
Hanger tension	162	142	148	140	224
M_R/M_T	−1%	+25%	−22%	+10%	+9%
M_G/M_T	+101%	+75%	+122%	+90%	+91%
Final deflection coefficients	−0.0161	−0.0177	−0.0025	+0.0214	+0.0360

The total moment

$$M_T = M' - H\ (y - y') \ldots\ldots (39)$$

is distributed to the girder and arch rib thus:

$$M_G = M' - M_I + H\ y' \ldots\ldots (40a)$$

and

$$M_R = M_I - H\ y \ldots\ldots (40b)$$

in which M_I is the simple beam moment due to interaction forces only; the moment is considered positive when producing tension in the lower fiber. The moments M_T, M_G, and M_R, and also the computed value of M_T using the author's value of H as given in his final design, are plotted in Fig. 13. The effect of the tie girder curvature on interactions again has been found to be quite negligible, varying from − 2% to + 1.5%.

This concludes a study of the various factors entering into the analysis of the St. Georges Tied Arch Span. After this step the analysis is shifted to an examination of the author's values. It has been found that, assuming the author's influence lines for H as being correct and using the distribution of bending moment, $M_T = M' - H\ (y - y')$, to rib and girder as 7% and 93%,

respectively, the resulting values of interactions would not be very different from the values shown in Table 4. Treating these interactions as loads on the isolated arch rib, and allowing neither elongation in the tie girder nor rib shortening, the value of the horizontal thrust becomes $H = 1{,}047$ lb.

Finally, all fourteen interactions for a unit load of 1,000 lb at point 7 were similarly determined by means of Eq. 18, using the given values of $H = 1{,}130$ lb. These interactions, treated as externally applied loads on the isolated arch rib, produce the horizontal thrust $H = 1{,}049$ lb only. Unless the assumption of the elastic properties of the structure was grossly in error, the writer does not believe that the value of the horizontal thrust for the load at point 7 could be greater than 1,049 lb. This limiting value of H is for the case of an arch rib receiving no support from the tie girder. Using the author's

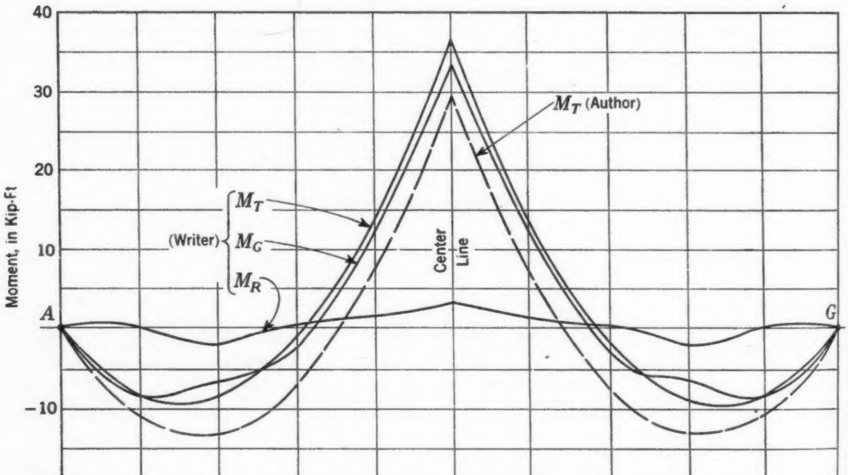


FIG. 13.—MOMENTS M_T, M_R, AND M_G

value of the horizontal thrust ($H = 1{,}130$ lb), and the resulting values of the interactions, the deflection curve of the arch rib was found to be quite different from the deflection curve of the tie girder. Furthermore, the deflection of the crown point of the arch was found to be negative (upward deflection with respect to original position of the rib). To move the unit load upward, however, the work must be done against gravity. Therefore, some other external force must be acting on the structure besides the unit load. This external force is the discrepancy between the writer's and the author's values of the horizontal thrust. With the value of $H = 1{,}130$ lb, the ends of the arch rib at the springing actually move inward with respect to their original position, which is contrary to the physical behavior of the structure.

No analysis was made by means of the coefficients of arch deflections for the other positions of the load. It is possible that the actual properties of the structure, complete data for which are not included in the paper, might in-

fluence somewhat the results of this analysis. The writer hopes that, in his concluding discussion, the author will give the missing information, as well as the magnitude of his interactions and deflections.

Conclusion.—This discussion may be summarized under four items, as follows:

(1) In the design of the St. Georges Tied Arch Span the author has compared the deflections of the tie girder, Table 2(*c*), with the deflections of the laterally unsupported arch rib, Table 2(*a*). From the foregoing discussion it is evident that the arch rib is practically fixed laterally, and the design should be based on the comparison of deflections of Tables 2(*a*) and 2(*b*). Therefore, the derivation of the author's formulas beyond Eq. 3 is not justified.

(2) The flitched-beam idea is not applicable for analysis of these structures, the component parts of which have no similarity to deflections for unit loads. The deflection of an arch rib, when laterally unsupported, is practically the same as the deflection of a simply supported beam of the same cross section. The deflection of the arch rib, when laterally supported, is just a fraction of the deflections of the laterally unsupported ribs. Furthermore, the deflection curves have no similarity.

(3) The great influence of longitudinal forces on the deflections of an arch rib must be emphasized. It is clearly evident from a comparison of Fig. 10 and Table 2(*a*) and 2(*b*).

(4) The function of the tie girder should not be overlooked. In the St. Georges Tied Arch Span the tie girder acts like a simple beam with an initial curvature. It could not act like an arch rib since the middle ordinate of curvature (6.8 ft) is less than the depth of the tie girder. Therefore, the horizontal deflection is not possible. If the middle ordinate were such as to permit the horizontal deflections, then both the tie girder and the arch rib would act like two simple beams, and the horizontal thrust would be equal to zero. This is plainly evident from Eq. 30 and Table 2(*a*).